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A stable cheapest finite element method for the Stokes problem and its application to the Stokes-Darcy-Brinkman interface problem

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김 시 환

Abstract

A stable cheapest finite element method for the Stokes problem and its application to the Stokes-Darcy-Brinkman interface problem

Sihwan Kim
The Interdisciplinary Program
in Computational Science and Technology
The Graduate School
Seoul National University

We introduce the pair of cheapest nonconforming finite element to approximate the Stokes equations in two and three dimension. The finite element space for the velocity field is composed of the P_1–nonconforming quadrilateral element or added by additional bubble functions based on $DSSY$ nonconforming finite space. The pressure field is approximated by the piecewise constant function. In two dimension case, we show that proposed two finite element pairs satisfy the discrete inf-sup condition uniformly. And we investigate the relationship between them. However, in the three dimensional case, we show that the proposed finite element pair satisfy the weak discrete inf-sup condition. Several numerical examples are shown to confirm the efficiency and reliability of the proposed method.

As the application, in two dimensional case, we discuss finite element methods for flows which are governed by the linear stationary Stokes problem on one part of the domain and by the Brinkman problem which can be reduced to Darcy problem in the rest of the domain. The solutions in the two domains are coupled by suitable interface conditions. We will introduce nonconforming finite element method to solve these coupled problem. For the Stokes-Brinkman problem, the velocity is approximated by P_1–nonconforming element, and pressure is approximated by piecewise constants. On the other hand, for the Stokes-Darcy problem, the velocity is approximated by P_1–nonconforming element and lowest order Raviart-Thomas element in the fluid region and porous medium, respectively. The pressure is still approximated by the piecewise constants. We show that the two coupled discrete problem satisfy the inf-sup condition. Moreover, we construct a new interpolation operator to derive *a priori* error estimate of the proposed finite element method. Several numerical examples are shown to confirm the efficiency and reliability of the proposed method.

Keywords : nonconforming finite element, P_1–nonconforming finite element, DSSY finite element, the Inf-sup condition, the incompressible Stokes equations, the Stokes-Darcy-Brinkman interface problem

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Chapter 1

The two-dimensional cheapest finite element method for the Stokes problem

1.1 Introduction

We are interested in the numerical approximation of the stationary incompressible Stokes problem. It has been well-known in fluid mechanics community that the lowest-degree conforming element $([P_1]^2, P_0)$ or $([Q_1]^2, P_0)$ produces numerically unstable solutions in the approximation of the pressure variable [21]. In particular Boland and Nicolaides [5, 6] fully investigate for the pair $([Q_1]^2, P_0)$. The above simple pair does not satisfy the discrete inf-sup condition. Several successful finite elements satisfying this condition have been proposed and used. For instance conforming finite element spaces [4, 16, 44, 46] including the $([P_2]^2, P_0)$ and $([P_2]^2, P_1)$ (the Taylor-Hood element) elements

[22, 27] and the MINI element [2].

By breaking the conformity of finite element space and relaxing the continuity across the inter-element interfaces instead of the throughout the interfaces, nonconforming finite element methods have been developed so that the discrete inf-sup condition is satisfied. [13] introduced the piecewise linear nonconforming finite element for the velocity and the piecewise constant element for the pressure $([P_1^{nc}]^2, P_0)$ as well as odd higher-degree nonconforming element pairs on triangular and tetrahedral meshes. This family has been proved to be stable for the Stokes problem and to give optimal orders of convergence. This $([P_1^{nc}]^2, P_0)$ pair has been regarded as one of the cheapest stable Stokes element pairs. A successful quadratic degree counterparts have been developed by Fortin and Soulie [18] and Fortin [17] in two and three dimensions, respectively. Such approaches to use nonconforming elements in rectangle-type of meshes have been tried by several researchers.

In many cases, it is also desirable to use quadrilateral elements rather than simplicial ones. In this direction Han [25] introduced a rectangular nonconforming finite element with five degrees of freedom in 1984 in approximating the velocity field while Rannacher and Turek [42] developed a simple rotated Q_1–element with four degrees of freedom in 1992. In 1999, Douglas *et al.* [15] modified the rotated Q_1–element and proposed a quadrilateral nonconforming element with four degrees of freedom so that the integral average and the barycenter value over each element interface are in agreement, which we call $DSSY$ element. Immediately, Cai *et al.* [10] applied this quadrilateral nonconforming element to solve the two and three dimensional stationary Stokes and Navier-Stokes equations with piecewise constant element pair and showed that this pair satisfies the discrete inf-sup condition.

In 2003, Park and Sheen [39] presented a P_1–nonconforming quadrilateral element to solve second-order elliptic problems which consists of linear polynomials only on each quadrilateral and the barycenter values DOF at the element interfaces. Moreover, Grajewski, Hron and Turek [23, 24] illuminated in detail the numerical behavior of this element with special emphasis on the treatment of Dirichlet boundary conditions and examined several numerical examples for the P_1–nonconforming finite element on quadrilateral meshes. In a similar direction, Hu and Shi [29] constructed a constrained nonconforming rotated Q_1 element and Mao *et al.* [36] developed a double set parameter element, which give us a different approach to obtain a nonconforming quadrilateral element of low order. It reduces the rotated Q_1 element to a simple bilinear element, which is equivalent to the P_1–nonconforming element if the mesh is rectangular.

Based on this element, several authors have tried to solve Stokes and Navier-Stokes equations by using P_1–nonconforming finite element and piecewise constant pair. Liu and Yan [34] and Hu, Man, Shi [28] proved the discrete inf-sup condition with the piecewise constant which removed one macro pressure for each macro rectangle that consists of four rectangles. For quadrilateral meshes, Park-Sheen-Shin [40] added one macro bubble velocity element based on the $DSSY$ nonconforming space on each macro quadrilateral which is the union of four quadrilaterals to propose a stable finite element pair.

In this paper, we propose two stable cheapest finite element pairs based on the P_1–nonconforming quadrilateral element and the piecewise constant element pair. Our direction to design cheapest element pairs is to modify velocity space minimally or to preserve pressure space as rich as possible by modifying it also minimally.

The first stable pair is obtained by removing the globally one-dimensional checkerboard pattern from the pressure space, while each component of the velocity fields is approximated by the P_1–nonconforming quadrilateral element. (For a conforming counterpart, see [21].) For the second stable pair, the P_1–nonconforming finite element space should be enriched by adding a globally one–dimensional $DSSY$-type (or Rannacher-Turek type) bubble space based on macro interior edges, with the simple piecewise constant pressure element unmodified.

Indeed, these two finite element pairs are closely related. We show that the velocity solutions obtained by these two finite element pairs are identical while the pressure solutions differ only by a term $\mathcal{O}(h)$ times the global discrete checkerboard pattern. Thus, the stability and optimal convergence results for one finite element pair are equivalent to those for the other.

The outline of this paper is organized as follows. In Section 2, the Stokes problem will be stated and the finite element pair will be defined. In Section 3, we will devote to check the discrete inf-sup condition for our proposed finite element pair by using the technique derived by Qin [41]. In Section 4, we will give the relationship between our two finite element pair. Finally, some numerical results are presented in Section 5.

1.2 The Stokes problem and the enrichment of velocity space

In this section we will introduce two pairs of stable nonconforming finite element spaces for the incompressible Stokes problem in two dimensions. We begin by examining the P_1-nonconforming quadrilateral element and the piecewise constant element. Suitable minimal modifications will be made so that uniform discrete inf-sup condition holds.

1.2.1 Notation and preliminaries

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a polygonal boundary and consider the following stationary Stokes problem:

$$
\begin{aligned}
-\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, && (1.1a)\\
\nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega, && (1.1b)\\
\mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega, && (1.1c)
\end{aligned}
$$

where $\mathbf{u} = (u_1, u_2)^T$ represents the velocity vector, p the pressure, $\mathbf{f} = (f_1, f_2)^T \in [H^{-1}(\Omega)]^2$ the body force, and $\nu > 0$ the viscosity. Set

$$L_0^2(\Omega) = \{q \in L^2(\Omega) \mid \int_\Omega q \; d\mathbf{x} = 0\}.$$

Here, and in what follows, we use the standard notations and definitions for the Sobolev spaces $[H^s(S)]^2$, and their associated inner products $(\cdot,\cdot)_{s,S}$, norms $||\cdot||_{s,S}$, and semi-norms $|\cdot|_{s,S}$. We will omit the subscripts s, S if $s = 0$ and $S = \Omega$. Also for boundary ∂S of S, the inner product in $L^2(\partial S)$ is denoted by $\langle\cdot,\cdot\rangle_S$. Then, the weak formulation of (1.1) is to seek a pair $(\mathbf{u}, p) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ such that

$$
\begin{aligned}
a(\mathbf{u},\mathbf{v}) - b(\mathbf{v}, p) &= (\mathbf{f},\mathbf{v}) \quad \forall\mathbf{v} \in [H_0^1(\Omega)]^2, && (1.2a)\\
b(\mathbf{u}, q) &= 0 \qquad\quad \forall q \in L_0^2(\Omega), && (1.2b)
\end{aligned}
$$

where the bilinear forms $a(\cdot,\cdot) : [H_0^1(\Omega)]^2 \times [H_0^1(\Omega)]^2 \to \mathbb{R}$ and $b(\cdot,\cdot) : [H_0^1(\Omega)]^2 \times L_0^2(\Omega) \to \mathbb{R}$ are defined by

$$a(\mathbf{u},\mathbf{v}) = \nu(\nabla\mathbf{u}, \nabla\mathbf{v}), \quad b(\mathbf{v}, q) = (\nabla\cdot\mathbf{v}, q).$$

Let $\mathcal{D} = \{\mathbf{v} \in [H_0^1(\Omega)]^2 \mid \nabla \cdot \mathbf{v} = 0\}$ denote the divergence–free subspace of $[H_0^1(\Omega)]^2$. Then the solution $\mathbf{u}$ of (1.2) lies in $\mathcal{D}$ and satisfies

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{D}. \tag{1.3}$$

1.2.2 Nonconforming finite element spaces

In order to highlight our approach to design new finite element methods, we shall restrict our attention to the case of $\Omega = [0,1]^2$. Let $(\mathcal{T}_h)_{0<h<1}$ be a family of uniform triangulation of Ω into disjoint squares Q_{jk} of size h for $j, k = 1, \cdots, N$ and $\overline{\Omega} = \bigcup_{j,k=1}^{N} \overline{Q}_{jk}$. $\mathcal{E}_h$ denotes the set of all edges in $\mathcal{T}_h$. Let N_Q and N_v^i be the number of elements and interior vertices, respectively.

The approximate space for velocity fields is based on the P_1–nonconforming quadrilateral element [9, 15, 39]. Set

$$\begin{aligned} \mathcal{NC}_0^h = \{v \in L^2(\Omega) \mid \ & v|_Q \in P_1(Q) \ \forall Q \in \mathcal{T}_h, v \text{ is continuous at the midpoint} \\ & \text{of each interior edge in } \mathcal{T}_h, \text{ and } v \text{ vanishes at the midpoint} \\ & \text{of each boundary edge in } \mathcal{T}_h\}. \end{aligned}$$

It is known that the pair of spaces $[\mathcal{NC}_0^h]^2 \times P^h$ cannot be used to solve the Stokes equations, as stated in the following theorem:

Theorem 1.2.1 ([38]). *Let $(\mathcal{T}_h)_{0<h<1}$ be a family of triangulations of Ω into rectangles and set*

$$\mathcal{C}^h = \{p_h \in P^h : \ b_h(\mathbf{v}_h, p_h) = 0 \quad \forall \mathbf{v}_h \in [\mathcal{NC}_0^h]^2\}.$$

Then $\dim(\mathcal{C}^h) = 1$*. Indeed, the elements $p_h \in \mathcal{C}^h$ are of global checkerboard pattern.*

We now try to stabilize $([\mathcal{NC}_0^h]^2, P^h)$ minimally so that the modified pairs fulfill the uniform inf-sup condition. Among several possible directions, we will consider the following cases:

1. Stabilization of P^h;

2. Stabilization of $[\mathcal{NC}_0^h]^2$.

For odd integers j and k, consider the macro element consist of $Q_{jk}, Q_{j,k+1}, Q_{j+1,k}, Q_{j+1,k+1}$. Then the one-dimensional subspace on the four quadrilaterals $\begin{bmatrix} Q_{j,k+1} & Q_{j+1,k+1} \\ Q_{j,k} & Q_{j+1,k} \end{bmatrix}$ spanned by

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} =: p_{JK}^{mc} \tag{1.4}$$

is the elementary checkerboard pattern. Let us associate for each odd indices J, K such the macro checkerboard pattern on Q_{JK}^M, and call it "p_{JK}^{mc}." Let $\mathcal{T}^M$ be the set of all macro element. Here, and in what follows, the boldface indices J and K are used to denote the odd integers only.

Denote by P_{mc}^h the space of such macro checkerboard patterns so that

$$P_{mc}^h = \left\{ q_h \in P^h \;\middle|\; q_h = \sum_{JK} a_{JK} p_{JK}^{mc} \right\} \quad \text{and} \quad W^h = P^h \ominus P_{mc}^h,$$

such that

$$P^h = P_{mc}^h \oplus W^h.$$

Instead of removing all the checkerboard macro elements from P^h, we will

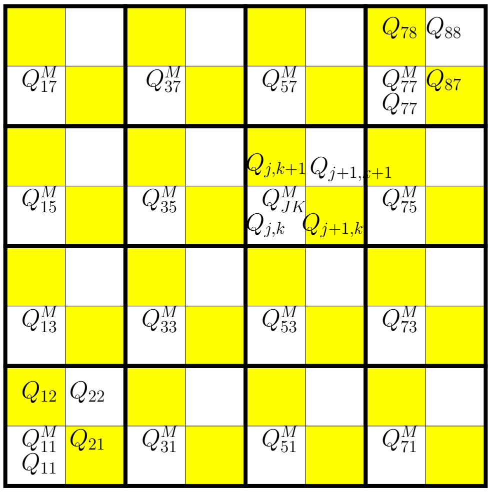


Figure 1.1: Macro elements: $Q^M_{JK} = Q_{j,k} \cup Q_{j,k+1} \cup Q_{j+1,k} \cup Q_{j+1,k+1}$

eliminate minimal possible such elements. For this, set

$$\mathcal{P}^h_{cf} = \mathcal{P}^h_c \oplus W^h, \quad \text{where} \quad \mathcal{P}^h_c = \left\{ q_h \in P^h \;\middle|\; q_h = \sum_{JK} a_{JK} p^{mc}_{JK}, \; \sum_{JK} a_{JK} = 0 \right\}.$$

Also set

$$\mathcal{P}^h_{cf'} = \mathcal{P}^h_{c'} \oplus W^h, \quad \text{where} \quad \mathcal{P}^h_{c'} = P^h_{mc} \ominus \left\{ q_h \in P^h \;\middle|\; q_h = \sum_{JK} p^{mc}_{JK} \right\}.$$

Notice that $\mathcal{P}^h_{cf}$ and $\mathcal{P}^h_{cf'}$ are subspaces of P^h of dimension $N_Q - 2$, while both are orthogonal to $\sum_{JK} p^{mc}_{JK}$; hence they are identical.

Finally, the suitable finite element space the pair to solve Stokes problem is $([\mathcal{NC}^h_0]^2, \mathcal{P}^h_{cf})$. Another way is to add a bubble function to the P_1-nonconforming element by using another quadrilateral nonconforming bubble function [15, 42]. Then, the discrete problem of the Stokes problem is solvable in the modified nonconforming finite element spaces. In this paper, we will study both methods.

Remark 1.2.2. *The dimension of the pair of spaces* $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$ *is* $2N_v^i + N_Q - 2$.

1.2.3 The enrichment of velocity space

In order to ensure the unique solvability, we add a single global bubble function to the nonconforming space $[\mathcal{NC}_0^h]^2$. On a reference domain $\widehat{Q} := [-1,1]^2$, the $DSSY$ nonconforming element spaces are defined by

$$DSSY(\widehat{Q}) = \text{Span}\{1, \widehat{x}, \widehat{y}, \ell_k(\widehat{x}) - \ell_k(\widehat{y})\},$$

where

$$\ell_k(t) = \begin{cases} t^2 - \frac{5}{3}t^4, & k = 1, \\ t^2 - \frac{25}{6}t^4 + \frac{7}{2}t^6, & k = 2. \end{cases}$$

Let $F_Q : \widehat{Q} \to Q$ be a bijective bilinear transformation from the reference domain onto the physical element Q. Then define

$$DSSY(Q) = \left\{ \widehat{v} \circ F_Q^{-1} \;\middle|\; \widehat{v} \in DSSY(\widehat{Q}) \right\} \tag{1.5}$$

The main characteristic of $DSSY(Q)$ is the edge-mean-value property:

$$\oint_E \psi \, ds = \psi(\text{midpoint of E}) \quad \forall \psi \in DSSY(Q), \tag{1.6}$$

where $\oint_E$ denotes $\frac{1}{|E|}\int_E$. The $DSSY$ nonconforming finite element spaces [9, 10, 15] are defined by

$$\begin{aligned} DSSY_0^h = \{ v \in L^2(\Omega) \mid v_j := v|_{Q_j} \in DSSY(Q_j) \; \forall j = 1, \cdots, N_Q; \\ v \text{ is continuous at the midpoint of each interior edge} \\ \text{and vanishes at the midpoint of each boundary edge in } \mathcal{T}_h \}. \end{aligned}$$

For each $Q_{JK}^M \in \mathcal{T}^M$, define $\boldsymbol{\psi}_{Q_{JK}^M} \in [DSSY_0^h]^2$ by

$$\operatorname{supp}(\boldsymbol{\psi}_{Q_{JK}^M}) \subset \overline{Q}_{JK}^M,$$

and the integral averages over the edges in $\mathcal{T}_h$ vanish except

$$\oint_{\partial Q_{j,k}\cap\partial Q_{j+1,k}} \boldsymbol{\psi}_{Q_{JK}^M}\ ds = \boldsymbol{\nu}, \qquad \oint_{\partial Q_{j,k+1}\cap\partial Q_{j+1,k+1}} \boldsymbol{\psi}_{Q_{JK}^M}\ ds = -\boldsymbol{\nu}.$$

where $\boldsymbol{\nu}$ denotes the unit outward normal vector of $Q_{j,k}$ on the edge $\partial Q_{j,k} \cap \partial Q_{j+1,k}$. From now on, $[\mathcal{NC}_0^h]^2$ will be enriched by adding global macro bubble functions.

Then, set

$$\mathbf{V}^h = [\mathcal{NC}_0^h]^2 \ \oplus\ \mathbf{B}^h,$$

where $\mathbf{B}^h = \operatorname{Span}\left\{ \sum_{Q_{JK}^M \in \mathcal{T}^M} \boldsymbol{\psi}_{Q_{JK}^M} \right\}$ is the space of bubble functions of dimension one.

Remark 1.2.3. *The dimension of the pair of spaces $(\mathbf{V}^h, P^h)$ is $2N_v^i + N_Q$.*

Thus, our finite element space to solve Stokes problem is one of $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$ or $(\mathbf{V}^h, P^h)$. We will demonstrate the pair of spaces $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$ in detail. The pair of spaces $(\mathbf{V}^h, P^h)$ can be understood with a slightly modification of $([\mathcal{NC}_0^h]^2, P^h)$. Now define the discrete weak formulation of (1.2) to find a pair $(\mathbf{u}_h, p_h) \in [\mathcal{NC}_0^h]^2 \times \mathcal{P}_{cf}^h$ such that

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) - b_h(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in [\mathcal{NC}_0^h]^2, && (1.7a)\\ b_h(\mathbf{u}_h, q_h) &= 0 \qquad\quad\ \forall q_h \in \mathcal{P}_{cf}^h, && (1.7b) \end{aligned}$$

where the discrete bilinear forms $a_h(\cdot,\cdot) : [\mathcal{NC}_0^h]^2 \times [\mathcal{NC}_0^h]^2 \to \mathbb{R}$ and $b_h(\cdot,\cdot)$:

$[\mathcal{NC}_0^h]^2 \times \mathcal{P}_{cf}^h \to \mathbb{R}$ are defined in the standard fashion:

$$a_h(\mathbf{u}, \mathbf{v}) = \nu \sum_{j=1}^{N_Q} (\nabla \mathbf{u}, \nabla \mathbf{v})_{Q_j} \quad \text{and} \quad b_h(\mathbf{v}, q) = \sum_{j=1}^{N_Q} (\nabla \cdot \mathbf{v}, q)_{Q_j}.$$

As usual, let $|\cdot|_{1,h}$ denote the (broken) energy semi-norm given by

$$|\mathbf{v}|_{1,h} = \sqrt{a_h(\mathbf{v}, \mathbf{v})},$$

which is equivalent to $\|\cdot\|_1$ on $[\mathcal{NC}_0^h]^2$. Also, denote by $\|\cdot\|_{m,h}$ and $|\cdot|_{m,h}$ the usual mesh-dependent norm and semi-norm:

$$\|\mathbf{v}\|_{m,h} = \left[\sum_{Q \in \mathcal{T}_h} \|\mathbf{v}\|^2_{H^m(Q)} \right]^{1/2} \quad \text{and} \quad |\mathbf{v}|_{m,h} = \left[\sum_{Q \in \mathcal{T}_h} |\mathbf{v}|^2_{H^m(Q)} \right]^{1/2},$$

respectively. Let $\mathcal{D}^h$ denote the divergence–free subspace of $[\mathcal{NC}_0^h]^2$ to $\mathcal{P}_{cf}^h$, *i.e.*,

$$\mathcal{D}^h = \{\mathbf{v}_h \in [\mathcal{NC}_0^h]^2 \mid b_h(\mathbf{v}_h, q_h) = 0, \ \forall q_h \in \mathcal{P}_{cf}^h\}. \tag{1.8}$$

Then the solution $\mathbf{u}_h$ of (1.7) lies in $\mathcal{D}^h$ and satisfies

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathcal{D}^h. \tag{1.9}$$

1.3 The inf-sup condition

In this section we will show that the pair of spaces $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$ and $(\mathbf{V}^h, P^h)$ satisfy the inf-sup condition. First we focus on the pair of spaces $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$. There are some useful techniques for checking the inf-sup condition[21]. In particular, we will use the technique called subspace theorem derived by Qin [41] for checking the inf-sup condition. The main idea of the theorem is to find two subspaces in each of the velocity and pressure spaces, and to check whether the four subspaces satisfy some conditions.

To this end, let $\mathbf{Z}^h$ denote the discrete divergence–free subspace of $[\mathcal{NC}_0^h]^2$ to $\mathcal{P}_c^h$, that is,

$$\mathbf{Z}^h = \left\{ \mathbf{v}_h \in [\mathcal{NC}_0^h]^2 \;\middle|\; b_h(\mathbf{v}_h, \widetilde{q}_h) = 0, \;\forall \widetilde{q}_h \in \mathcal{P}_c^h \right\}.$$

Similarly, we define

$$\mathbf{X}_b^h = \left\{ \mathbf{v}_{bh} \in [H_0^1(\Omega)]^2 \;\middle|\; \mathbf{v}_{bh}|_Q \text{ is bilinear}, \;\forall Q \in \mathcal{T}_h \right\}, \tag{1.10}$$

and $\mathbf{Z}_b^h$ denote the discrete divergence–free subspace of $\mathbf{X}_b^h$ to $\mathcal{P}_c^h$, that is,

$$\mathbf{Z}_b^h = \left\{ \mathbf{v}_{bh} \in \mathbf{X}_b^h \;\middle|\; b_h(\mathbf{v}_{bh}, \widetilde{q}_h) = 0, \;\forall \widetilde{q}_h \in \mathcal{P}_c^h \right\}.$$

Also let $[\widetilde{\mathcal{NC}_0^h}]^2$ be the subspace of $[\mathcal{NC}_0^h]^2$ defined by

$$[\widetilde{\mathcal{NC}_0^h}]^2 = \left\{ \mathbf{v}_h \in [\mathcal{NC}_0^h]^2 \;\middle|\; \mathbf{v}_h = \sum_{\Gamma^M \in \mathcal{E}_{2h}} \begin{pmatrix} a_{\Gamma^M} \\ b_{\Gamma^M} \end{pmatrix} \Psi_{\Gamma^M}, \quad \begin{pmatrix} a_{\Gamma^M} \\ b_{\Gamma^M} \end{pmatrix} \in \mathbb{R}^2 \right\} \tag{1.11}$$

where Ψ_{Γ^M} is the basis function associated with the midpoint of the macro edge $\Gamma^M \in \mathcal{E}_{2h}$ as in Figure 1.2. Notice that $\dim([\widetilde{\mathcal{NC}_0^h}]^2) = N_v^i - 1$.

Theorem 1.3.1 ([41])**.** *Given $(\mathbf{V}^h, P^h)$, let $\mathbf{V}_1$ and $\mathbf{V}_2$ be two subspaces of $\mathbf{V}^h$ and P_1 and P_2 be two subspaces of P^h. Let the following three conditions hold:*

1. $P^h = P_1 + P_2$;

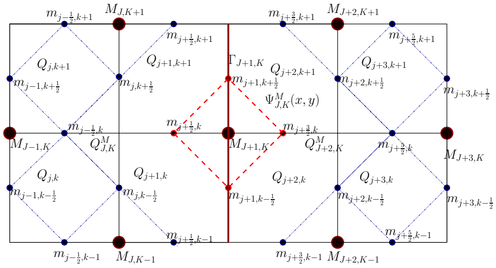


Figure 1.2: The basis function $\Psi_{\Gamma^M} \in [\mathcal{NC}_0^h]^2$, associated with the macro edge $\Gamma^M = \Gamma^M_{J+1,K}$, takes the value 1 along the four line segments joining the midpoints $m_{j+\frac{3}{2},k}, m_{j+1,k+\frac{1}{2}}, m_{j+\frac{1}{2},k}$, and $m_{j+1,k-\frac{1}{2}}$, and the value 0 at any other midpoints m's shown in the figure. $M_{J+1,K}$ denotes the midpoint of the macro edge $\Gamma^M_{J+1,K}$, the common edge of the two macro elements $Q^M_{J,K}$ and $Q^M_{J+2,K}$, with $Q^M_{J,K} = Q_{j,k} \cup Q_{j,k+1} \cup Q_{j+1,k} \cup Q_{j+1,k+1}$ and $Q^M_{J+2,K} = Q_{j+2,k} \cup Q_{j+2,k+1} \cup Q_{j+3,k} \cup Q_{j+3,k+1}$.

2. *there exist $\alpha_j > 0, j = 1, 2$, independent of h, such that*

$$\sup_{\mathbf{v}_j \in \mathbf{V}_j} \frac{b_h(\mathbf{v}_j, q_j)}{|\mathbf{v}_j|_{1,h}} \geq \alpha_j \|q_j\|_{0,\Omega}, \quad \forall q_j \in P_j,$$

3. *there exist $\beta_j, \geq 0, j = 1, 2$, such that*

$$|b_h(\mathbf{v}_k, q_\ell)| \leq \beta_k |\mathbf{v}_k|_{1,h} \|q_\ell\|_{0,\Omega}, \quad \forall \mathbf{v}_k \in \mathbf{V}_k \text{ and } \forall q_\ell \in P_\ell,\ k, \ell, = 1, 2; k \neq \ell,$$

with

$$\beta_1 \beta_2 \leq \alpha_1 \alpha_2.$$

Then $(\mathbf{V}^h, P^h)$ satisfies the inf-sup condition with the inf-sup constant depending only on $\alpha_1, \alpha_2, \beta_1, \beta_2$.

Theorem 1.3.2. *The pair of finite element spaces $([\mathcal{NC}_0^h]^2, \mathcal{P}^h_{cf})$ satisfies the inf-sup condition, that is, there exists a positive constant β independent of h.*

Proof. Let $\mathbf{V}_1 = \mathbf{Z}^h,\ \mathbf{V}_2 = [\widetilde{\mathcal{NC}_0^h}]^2$ and $P_1 = W^h,\ P_2 = \mathcal{P}_c^h$. Obiously, $\mathbf{V}_j$ and $P_j,\ j = 1, 2$ are subspaces of $[\mathcal{NC}_0^h]^2$ and $\mathcal{P}_{cf}^h$, respectively so condition (1) holds. Moreover, Lemma 1.3.3 and Lemma 1.3.4 imply condition (2) hold. Since $b_h(\mathbf{v}_1, q_2) = 0$ holds for any $\mathbf{v}_1 \in \mathbf{V}_1$ and any $q_2 \in P_2$, so $\beta_1 = 0$. Consequently, condition (3) holds. Hence by Theorem 1.3.1, $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$ satisfies the inf-sup condition. $\square$

Lemma 1.3.3. *The pair of finite element spaces $(\mathbf{Z}^h, W^h)$ satisfies the inf-sup condition, that is, there exists a positive constant β independent of h.*

Proof. We begin with invoking [6] that the pair of spaces $(\mathbf{Z}_b^h, W^h)$ satisfies the uniform inf-sup condition, that is, there exists a positive constant β independent of h such that

$$\sup_{\mathbf{v}_{bh} \in \mathbf{Z}_b^h} \frac{b_h(\mathbf{v}_{bh}, q_h)}{|\mathbf{v}_{bh}|_{1,h}} \geq \beta \|q_h\|_{0,\Omega} \quad \forall q_h \in W^h. \tag{1.12}$$

We can state an equivalent formulation of (1.12) as (cf.[21], pp. 118): For any $q_h \in W^h$, there is a function $\mathbf{v}_{bh} \in \mathbf{Z}_b^h$ such that

$$b_h(\mathbf{v}_{bh}, q_h) = \|q_h\|_{0,\Omega}^2, \tag{1.13a}$$

$$|\mathbf{v}_{bh}|_{1,\Omega} \leq \frac{1}{\beta} \|q_h\|_{0,\Omega}. \tag{1.13b}$$

For each $Q \in \mathcal{T}_h$, we define the local interpolation operator $\mathbf{\Pi}_Q : \left[H^2(Q)\right]^2 \cap \mathbf{X}_b^h(Q) \to \left[\mathcal{NC}_0^h(Q)\right]^2$ which satisfies the following conditions:

$$\mathbf{\Pi}_Q \mathbf{v}_{bh}(M_k) = \frac{\mathbf{v}_{bh}(V_{k-1}) + \mathbf{v}_{bh}(V_k)}{2}, \tag{1.14}$$

where V_{k-1} and V_k are the two vertices of the edge E_k with the midpoint M_k. For each $Q \in \mathcal{T}_h$, the edge-midpoint values defined by (1.14) determine a linear polynomial on Q uniquely. We define $\mathbf{\Pi}_h \mathbf{v}_{bh} \in [\mathcal{NC}_0^h]^2$ as the global interpolation operator given by

$$\mathbf{\Pi}_h \mathbf{v}_{bh}|_Q = \mathbf{\Pi}_Q \mathbf{v}_{bh}.$$

It is readily seen that $b_h(\mathbf{\Pi}_h \mathbf{v}_{bh}, q_h') = 0$ if $b_h(\mathbf{v}_{bh}, q_h') = 0$ for all $q_h' \in \mathcal{P}_c^h$. if $\mathbf{v}_{bh} \in \mathbf{Z}_b^h$ satisfying (1.13), if we consider $\mathbf{\Pi}_h \mathbf{v}_{bh} \in [\mathcal{NC}_0^h]^2$ then we have $\mathbf{\Pi}_h \mathbf{v}_{bh} \in \mathbf{Z}^h$ and

$$b_h(\mathbf{\Pi}_h \mathbf{v}_{bh}, q_h) = b_h(\mathbf{v}_{bh}, q_h) = \|q_h\|_{0,\Omega}^2. \tag{1.15}$$

By Young's inequality, definition of interpolation operator $\mathbf{\Pi}_h$ and (1.13b), we easily prove that

$$|\mathbf{\Pi}_h \mathbf{v}_{bh}|_{1,h} \le C|\mathbf{v}_{bh}|_{1,\Omega} \le \frac{C}{\beta}\|q_h\|_{0,\Omega}, \tag{1.16}$$

where constant C is independent of mesh size h. Therefore, (1.15) and (1.16) implies the desired assertion. □

Lemma 1.3.4. *The pair of finite element spaces $([\widetilde{\mathcal{NC}}_0^h]^2, \mathcal{P}_c^h)$ satisfies the inf-sup condition, that is, there exists a positive constant β independent of h.*

Proof. Due to Lemma 3.1 in [40], $([\widetilde{\mathcal{NC}}_0^h]^2, P^{2h})$ satisfies the uniform inf-sup condition, that is, there exists a positive constant β independent of h such that

$$\sup_{\overline{\mathbf{v}}_h \in [\widetilde{\mathcal{NC}}_0^h]^2} \frac{b_h(\overline{\mathbf{v}}_h, \overline{q}_h)}{|\overline{\mathbf{v}}_h|_{1,h}} \ge \beta \|\overline{q}_h\|_{0,\Omega} \quad \forall \overline{q}_h \in P^{2h}. \tag{1.17}$$

Let $q_h = \sum_{JK} \alpha_{JK} p_{JK}^{mc} \in \mathcal{P}_c^h$ be arbitrary. Consider $\overline{q}_h = \sum_{JK} \alpha_{JK} p_{JK} \in P^{2h}$.

Then there exist $\overline{\mathbf{v}}_h = \sum_{\Gamma^M \in \mathcal{E}_{2h}} \begin{pmatrix} a_{\Gamma^M} \\ b_{\Gamma^M} \end{pmatrix} \Psi_{\Gamma^M} \in [\widetilde{\mathcal{NC}}_0^h]^2$ such that (1.17) holds.

From given $\overline{\mathbf{v}}_h$, we define $\mathbf{v}_h \in [\widetilde{\mathcal{NC}}_0^h]^2$ as follows:

$$\mathbf{v}_h = -\sum_{\Gamma^M \in \mathcal{E}_{2h}} \begin{pmatrix} b_{\Gamma^M} \\ a_{\Gamma^M} \end{pmatrix} \Psi_{\Gamma^M}.$$

Then the following three equalities are obvious:

$$\|q_h\|_{0,\Omega} = \|\overline{q}_h\|_{0,\Omega}, \tag{1.18a}$$

$$|\mathbf{v}_h|_{1,h} = |\overline{\mathbf{v}}_h|_{1,h}, \tag{1.18b}$$

$$b_h(\mathbf{v}_h, q_h) = b_h(\overline{\mathbf{v}}_h, \overline{q}_h). \tag{1.18c}$$

From (1.18), we get the inf-sup condition for the pair of spaces $([\widetilde{\mathcal{NC}}_0^h]^2, \mathcal{P}_c^h)$ by using (1.17). □

Lemma 1.3.5. $(\mathbf{B}^h, \mathcal{C}^h)$ *satisfies the inf-sup condition, that is, there exists a positive constant* β *independent of* h.

Proof. Let $q_h \in \mathcal{C}^h$ be given. Since $\dim(\mathcal{C}^h) = 1$, q_h can be represented by

$$q_h = \alpha Alt_h,$$

where, $Alt_h = \sum_{JK} p_{JK}^{mc}$. Then, consider $\boldsymbol{\psi} \in \mathbf{B}^h$ with coefficient one. A direct calculation using the divergence theorem gives

$$b_h(\boldsymbol{\psi}, Alt_h) = \frac{1}{h}. \tag{1.19}$$

It remains to compute $\|q_h\|_{0,\Omega}$ and $|\boldsymbol{\psi}|_{1,h}$. For the first, we easily check that

$$\|q_h\|_{0,\Omega} = |\alpha| \tag{1.20}$$

and for the second, we notice that $|\boldsymbol{\psi}|_{1,Q}$ does not depend on the mesh size h of Q, since it is of two dimension. Thus, there is a constant C_1 independent of h such that

$$|\boldsymbol{\psi}|_{1,h}^2 = \sum_{Q\in\mathcal{T}_h} \int_Q |\nabla \boldsymbol{\psi}|^2 \, d\mathbf{x} = C_1 N^2. \tag{1.21}$$

Hence, we get $|\boldsymbol{\psi}|_{1,h} = CN$, where $C = \sqrt{C_1}$. Finally using (1.20)-(1.21) in (1.19) gives the inf-sup constant $\beta = 1/C$. This completes the proof. □

Remark 1.3.6. *The inf-sup condition for* $(\mathbf{V}^h, P^h)$ *can be proved in a similar way. Let* $\mathbf{V}_1 = [\mathcal{NC}_0^h]^2$, $\mathbf{V}_2 = \mathbf{B}^h$ *and* $P_1 = \mathcal{P}_{cf}^h$, $P_2 = \mathcal{C}^h$. *Since* $P^h = \mathcal{P}_{cf}^h \oplus \mathcal{C}^h$, *Condition* (1) *holds. Moreover, Theorem 1.3.2 and Lemma 1.3.5 imply Condition* (2) *hold. Finally,* $b_h(\mathbf{v}_1, q_2) = 0$ *holds for any* $\mathbf{v}_1 \in \mathbf{V}_1$ *and any* $q_2 \in P_2$ *by Theorem 1.2.1, so* $\beta_1 = 0$. *Consequently, Condition* (3) *holds. Hence,* $(\mathbf{V}^h, P^h)$ *satisfies the inf-sup condition.*

In contrast to the $O(h)$–dependent inf-sup constant of bilinear and piecewise constant finite element pairs [5, 6], our two proposed nonconforming finite elements satisfy the uniform inf-sup condition at least on square meshes. To confirm theoritical analysis, we give the numerical result for the inf-sup constant[35] in Table 1.1.

Define an interpolation operator $R_h : [H^2(\Omega)]^2 \cap [H_0^1(\Omega)]^2 \to [\mathcal{NC}_0^h]^2$ and

h	β_1	Order	β_2	Order	β_3	Order
1/4	4.9642E-01	-	4.9560E-01	-	5.0000E-01	-
1/8	2.8605E-01	0.78	4.6791E-01	0.08	4.6746E-01	0.09
1/16	1.5029E-01	0.93	4.4415E-01	0.07	4.5296E-01	0.04
1/32	7.6544E-02	0.97	4.2863E-01	0.05	4.4526E-01	0.02
1/64	3.8562E-02	0.99	4.1864E-01	0.03	4.4051E-01	0.02

Table 1.1: Estimation of the inf-sup constants $\beta_j, j = 1, 2, 3$, for the three finite element pairs $(\mathbf{X}_b^h, \mathcal{P}_{cf}^h)$, $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$, and $(\mathbf{V}^h, P^h)$.

$S_h : H^1(\Omega) \cap L_0^2(\Omega) \to \mathcal{P}_{cf}^h$ by

$$R_h\mathbf{w}(M_k^Q) = \frac{\mathbf{w}(V_{k-1}^Q) + \mathbf{w}(V_k^Q)}{2} \qquad \forall Q \in \mathcal{T}_h, \tag{1.22}$$

$$(S_h q, z) = (q, z) \qquad \forall z \in \mathcal{P}_{cf}^h \tag{1.23}$$

where V_{k-1}^Q and V_k^Q are the two vertices of the edge E_k^Q with the midpoint M_k^Q for each $Q \in \mathcal{T}_h$.

Since R_h and S_h reproduce linear and constant functions on each element $Q_j \in \mathcal{T}_h$ and macro element Q_{JK}^M, respectively, the standard polynomial approximation results imply that

$$\begin{aligned}\|\mathbf{v} - R_h\mathbf{v}\| + h\left(\sum_{Q_j\in\mathcal{T}_h} |\mathbf{v} - R_h\mathbf{v}|_{1,j}^2\right)^{1/2} + h^2\left(\sum_{Q_j\in\mathcal{T}_h} |\mathbf{v} - R_h\mathbf{v}|_{2,j}^2\right)^{1/2} \qquad (1.24a)\\
+h^{1/2}\left(\sum_{Q_j\in\mathcal{T}_h} ||\mathbf{v} - R_h\mathbf{v}||_{L^2(\partial Q_j)}^2\right)^{1/2} \le Ch^2\|\mathbf{v}\|_2, \quad \forall \mathbf{v} \in [H^2(\Omega)]^2 \cap [H_0^1(\Omega)]^2,\end{aligned}$$

$$\|q - S_h q\|_{0,\Omega} \le Ch\|q\|_1, \quad \forall q \in H^1(\Omega) \cap L_0^2(\Omega). \tag{1.24b}$$

Theorem 1.3.7. *Assume that* (1.1) *is* H^2*–regular. Let* $(\mathbf{u}, p)$ *and* $(\mathbf{u}_h, p_h)$ *be the solutions of* (1.2) *and* (1.7), *respectively. Then the following optimal-order* *error estimate holds:*

$$\|\mathbf{u}-\mathbf{u}_h\|_0 + h\,[\|\mathbf{u}-\mathbf{u}_h\|_1 + \|p-p_h\|] \le Ch^2(\|\mathbf{u}\|_2 + \|p\|_0). \tag{1.25}$$

1.4 The comparison $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$ and $(\mathbf{V}^h, P^h)$

In this section, we will compare the two nonconforming finite element space pairs $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$ and $(\mathbf{V}^h, P^h)$. For the pair of spaces $(\mathbf{V}^h, P^h)$, we have the following discrete weak formulation:

Find a pair $(\mathbf{u}_h', p_h') \in \mathbf{V}^h \times P^h$ *such that*

$$a_h(\mathbf{u}_h', \mathbf{v}_h') - b_h(\mathbf{v}_h', p_h') = (\mathbf{f}, \mathbf{v}_h') \quad \forall \mathbf{v}_h' \in \mathbf{V}^h, \tag{1.26a}$$

$$b_h(\mathbf{u}_h', q_h') = 0 \quad \forall q_h' \in P^h. \tag{1.26b}$$

Let $\mathcal{D}'^h$ denote the divergence–free subspace of $\mathbf{V}^h$ to P^h, *i.e.*,

$$\mathcal{D}'^h = \{\mathbf{v}_h' \in \mathbf{V}^h \mid b_h(\mathbf{v}_h', q_h') = 0,\ \forall q_h' \in P^h\}. \tag{1.27}$$

Then the solution $\mathbf{u}_h'$ of (1.26) lies in $\mathcal{D}'^h$ and satisfies

$$a_h(\mathbf{u}_h', \mathbf{v}_h') = (\mathbf{f}, \mathbf{v}_h') \quad \forall \mathbf{v}_h' \in \mathcal{D}'^h. \tag{1.28}$$

The following lemma implies that the spaces (1.8) and (1.27) are equivalent, that is, our two proposed nonconforming finite element space pairs $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$ and $(\mathbf{V}^h, P^h)$ give the identical solution for velocity.

Lemma 1.4.1. *The spaces* $\mathcal{D}^h$ *and* $\mathcal{D}'^h$ *defined by* (1.8) *and* (1.27)*, respectively, are equivalent.*

Proof. Let $\mathbf{v}_h \in \mathcal{D}^h$ be given. Since $q_h' \in \mathrm{Span}\{\mathcal{P}_{cf}^h \oplus \mathcal{C}^h\}$ and by Theorem 1.2.1, we get $b_h(\mathbf{v}_h, q_h') = 0$. This implies $\mathbf{v}_h \in \mathcal{D}'^h$, so $\mathcal{D}^h \subset \mathcal{D}'^h$. It remains to prove $\mathcal{D}'^h \subset \mathcal{D}^h$. Let $\mathbf{v}'_h = \mathbf{w}_h + \mathbf{b}_h \in \mathcal{D}'_h$ be given , where $\mathbf{w}_h \in [\mathcal{NC}_0^h]^2$ and $\mathbf{b}_h \in \mathbf{B}^h$. In particular, if we consider $q'_h \in \mathcal{C}^h$, then $b_h(\mathbf{v}'_h, q'_h) = 0$ implies $\mathbf{b}_h \equiv \mathbf{0}$. Thus for any $q'_h \in P^h$, $b_h(\mathbf{w}_h, q'_h) = b_h(\mathbf{w}_h, q_h) = 0$ for any $q_h \in \mathcal{P}_{cf}^h$ by Theorem 1.2.1. This implies $\mathcal{D}'^h \subset \mathcal{D}^h$. This completes the proof. □

Since $\mathbf{u}_h \equiv \mathbf{u}'_h$ by Lemma 1.4.1, the difference of discrete weak formulation (1.7a) and (1.26a) gives

$$b_h(\mathbf{v}_h, p'_h - p_h) = 0, \qquad \forall \mathbf{v}_h \in [\mathcal{NC}_0^h]^2.$$

By Theorem 1.2.1, $p'_h - p_h \in \mathcal{C}^h$, that is, p'_h can be represented by

$$p'_h = p_h + \alpha Alt_h, \qquad \alpha \in \mathbb{R}.$$

Taking $p'_h = p_h + \alpha Alt_h$ and $\mathbf{v}'_h = \boldsymbol{\psi}_h \in \mathbf{B}^h$ with coefficient one in (1.26a), we obtain

$$\begin{aligned} \alpha b_h(\boldsymbol{\psi}_h, Alt_h) &= a_h(\mathbf{u}_h, \boldsymbol{\psi}_h) - (\mathbf{f}, \boldsymbol{\psi}_h) - b_h(\boldsymbol{\psi}_h, p_h), \\ &= \nu \sum_{j=1}^{N_Q} (\nabla \mathbf{u}_h, \nabla \boldsymbol{\psi}_h)_{Q_j} - (\mathbf{f}, \boldsymbol{\psi}_h) - b_h(\boldsymbol{\psi}_h, p_h), \\ &= \nu \sum_{j=1}^{N_Q} (-\Delta \mathbf{u}_h, \boldsymbol{\psi}_h)_{Q_j} + \nu \left\langle \frac{\partial \mathbf{u}_h}{\partial \widehat{\mathbf{n}}}, \boldsymbol{\psi}_h \right\rangle_{\partial Q_j} - (\mathbf{f}, \boldsymbol{\psi}_h) \\ &- b_h(\boldsymbol{\psi}_h, p_h). \end{aligned} \tag{1.29}$$

Since the solution $\mathbf{u}_h$ is a piecewise linear polynomial, that is, $\mathbf{u}_h \in [\mathcal{NC}_0^h]^2$, the first term in (1.29) is equal to zero. And we easily check that the second and last term in (1.29) turn out to be zero by the characteristics of the space

$\mathbf{B}^h$. Invoking (1.19), we have

$$\alpha = -\frac{(\mathbf{f}, \boldsymbol{\psi}_h)}{b_h(\boldsymbol{\psi}_h, Alt_h)} = -h(\mathbf{f}, \boldsymbol{\psi}_h). \tag{1.30}$$

Hence, our two proposed nonconforming finite element space pairs $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$ and $(\mathbf{V}^h, P^h)$ are very closely related. First, the solution of velocity field are same by Lemma 1.4.1, and the solution p_h' is formulated by $p_h + \alpha Alt_h$, where p_h is the solution of (1.7) and α given by (2.22).

1.5 Numerical results

Now we illustrate a numerical example for the stationary Stokes problem on uniform meshes for the domain $\Omega = [0,1]^2$. Throughout the numerical study, we fix $\nu = 1$. We will bring the two numerical example examined in [40]. The source term $\mathbf{f}$ is generated by the choice of the exact solution.

$$\mathbf{u}(x,y) = (s(x)s'(y), -s(y)s'(x)), \qquad p(x,y) = \sin(2\pi x)f(y), \tag{1.31}$$

where $s(t) = \sin(2\pi t)(t^2 - t)$, $s'(t)$ denotes its derivative. The velocity $\mathbf{u}$ vanishes at $t = 0, 1$ so that $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$ and pressure p has mean value zero regardless of f.

Here we will give the numerical results for the pair $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$. The numerical results, omitted here, for the pair $(\mathbf{V}^h, P^h)$ behave quite similarly to those case for the pair $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$. The numerical results with $f(y) = \frac{1}{3-\tan^2 y}$ are shown in Table 1.2. We observe optimal order of convergence in both velocity and pressure variables. We try to another numerical example with $f(y) = \frac{1}{25-10\tan^2 y} + \frac{3}{10}$ in (1.31) which has a huge slope near the boundary on $y = 1$. Since the pressure changes rapidly on the boundary $y = 1$, convergence rate shows a poor approximation in coarse meshes in Table 1.3.

However, as the meshes get finer, optimal order convergence is observed as expected from the inf-sup condition.

h	$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}$	Order	$\|\mathbf{u}-\mathbf{u}_h\|_0$	Order	$\|p-p_h\|_0$	Order
1/4	1.5087E-0	-	2.1583E-1	-	2.2190E-1	-
1/8	8.1269E-1	0.8926	5.5033E-2	1.9715	1.4098E-1	0.6544
1/16	4.1360E-1	0.9745	1.3930E-2	1.9821	6.4738E-2	1.1229
1/32	2.0767E-1	0.9939	3.4936E-3	1.9954	3.2509E-2	0.9938
1/64	1.0394E-1	0.9985	8.7411E-4	1.9988	1.6411E-2	0.9862
1/128	5.1985E-2	0.9996	2.1857E-4	1.9997	8.2359E-3	0.9947
1/256	2.5994E-2	0.9999	5.4646E-5	1.9999	4.1222E-3	0.9985
1/512	1.2997E-2	1.0000	1.3661E-5	2.0000	2.0616E-3	0.9996
1/1024	6.4987E-3	1.0000	3.4154E-6	2.0000	1.0309E-3	0.9999

Table 1.2: Numerical result for uniform meshes with $f(y) = \frac{1}{3-\tan^2 y}$

h	$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}$	Order	$\|\mathbf{u}-\mathbf{u}_h\|_0$	Order	$\|p-p_h\|_0$	Order
1/4	1.5086E-0	-	2.1578E-1	-	1.7459E-1	-
1/8	8.1268E-1	0.8925	5.5016E-2	1.9716	1.1835E-1	0.5609
1/16	4.1360E-1	0.9744	1.3926E-2	1.9820	5.7158E-2	1.0501
1/32	2.0767E-1	0.9939	3.4938E-3	1.9950	3.6347E-2	0.6531
1/64	1.0394E-1	0.9985	8.7450E-4	1.9983	2.3178E-2	0.6491
1/128	5.1985E-2	0.9996	2.1872E-4	1.9993	1.3569E-2	0.7725
1/256	2.5994E-2	0.9999	5.4690E-5	1.9998	7.3091E-3	0.8925
1/512	1.2997E-2	1.0000	1.3673E-5	1.9999	3.7516E-3	0.9622
1/1024	6.4987E-3	1.0000	3.4183E-6	2.0000	1.8899E-3	0.9892

Table 1.3: Numerical result for uniform meshes with $f(y) = \frac{1}{25-10\tan^2 y} + \frac{3}{10}$

The following numerical results highlight the reliability of our proposed finite element space compared to the case of using the conforming bilinear element for the approximation of the velocity field. Recall that the pair of conforming finite element space combined with the piecewise constant element space $(\mathbf{X}_b^h, \mathcal{P}_{cf}^h)$ is unstable unless $\mathbf{f}$ is smooth enough:

Corollary 1.5.1 (Boland and Nicolaides, Cor. 6.1 in [6]). *For $\beta \in (0,1)$, there exists $\mathbf{f} \in [L^2(\Omega)]^2$ such that the pressure part of the numerical approximate* *solution to* (1.2) *by using* $(\mathbf{X}_b^h, \mathcal{P}_{cf}^h)$ *fulfills*

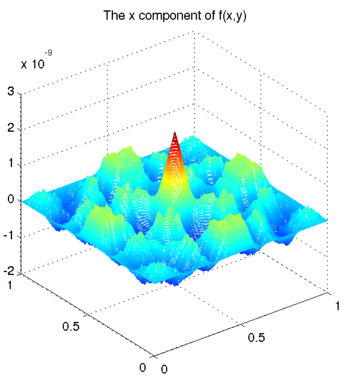
$$\|p - p_h\|_0 \geq Ch^\beta \|\mathbf{f}\|_0 \qquad \textit{for } h \leq h_\beta \tag{1.32}$$

for some $h_\beta > 0$, *independent of* h.

In our numerical results, $\beta = 0.3$ is fixed. The expected numerical results are shown in Table 1.4. To compare these results with those obtained by our proposed nonconforming method, we adopt the same $\mathbf{f}$. The numerical results in Table 1.5, quite different from those in Table 1.4, show the superiority of our nonconforming method.

Throughout this numerical example, we take the 4×4 Gauss quadrature rule for each rectangular element. The approximate data for $\mathbf{f}$ are obtained by following the proof of Thm. 6.1 in [6] at 4×4 Gauss points in each element of 512×512 mesh. The reference solutions used in our calculation are the numerical solutions using the DSSY element [15] with the 512×512 mesh. The graphs of the components of $\mathbf{f}$ are given Figure 1.3.

Remark 1.5.2. *It should be stressed that the degrees of freedom for* $(\mathbf{X}_b^h, \mathcal{P}_{cf}^h)$ *and* $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$ *are essentially identical, although the numerical results are quite different. Further investigations are needed to analyse the differences between the conforming bilinear element and the* P_1*–nonconforming element.*



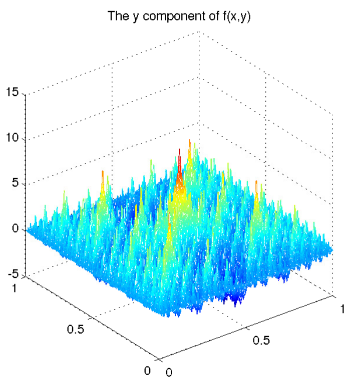


Figure 1.3: The graph of data $\mathbf{f}$

h	$\lvert\mathbf{u}_{ref} - \mathbf{u}_h\rvert_{1,h}$	order	$\lVert\mathbf{u}_{ref} - \mathbf{u}_h\rVert_0$	order	$\lVert p_{ref} - p_h\rVert_0$	order
1/4	2.8248E-2	-	1.8470E-3	-	7.2967E-2	-
1/8	1.6008E-2	0.8193	5.3114E-4	1.7981	5.6105E-2	0.3791
1/16	8.5909E-3	0.8980	1.4266E-4	1.8964	4.1920E-2	0.4205
1/32	4.4824E-3	0.9385	3.7531E-5	1.9265	3.1925E-2	0.3929
1/64	2.3084E-3	0.9573	9.6932E-6	1.9531	2.4932E-2	0.3567
1/128	1.1939E-3	0.9512	2.4703E-6	1.9722	1.9829E-2	0.3304
1/256	6.4542E-4	0.8874	6.2940E-7	1.9727	1.5938E-2	0.3152

Table 1.4: Numerical result for the pair of spaces $(\mathbf{X}_b^h, \mathcal{P}_{cf}^h)$ when $\beta = 0.3$

h	$\lvert\mathbf{u}_{ref} - \mathbf{u}_h\rvert_{1,h}$	order	$\lVert\mathbf{u}_{ref} - \mathbf{u}_h\rVert_0$	order	$\lVert p_{ref} - p_h\rVert_0$	order
1/4	2.8359E-2	-	1.8561E-3	-	4.9406E-2	-
1/8	1.7966E-2	0.6585	5.0224E-4	1.8858	2.6963E-2	0.8737
1/16	1.0379E-2	0.7916	1.3390E-4	1.9072	1.4305E-2	0.9144
1/32	5.6226E-3	0.8844	3.5144E-5	1.9298	7.5726E-3	0.9177
1/64	2.9406E-3	0.9351	9.0617E-6	1.9554	3.9235E-3	0.9486
1/128	1.5002E-3	0.9710	2.3029E-6	1.9763	1.9663E-3	0.9966
1/256	7.3601E-4	1.0274	5.7096E-7	2.0120	8.9372E-4	1.1376

Table 1.5: Numerical result for the pair of spaces $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$ when $\beta = 0.3$

Chapter 2

The application to the Stokes-Darcy-Brinkman interface problem

2.1 Introduction

The purpose of this paper is to discuss new finite element method of coupled Stokes-Darcy-Brinkman problem. The Brinkman model treats both Darcy's law and Stokes problem in a single form of PDE but with strongly discontinuous viscosity coefficient and zeroth-order term coefficient. When viscosity coefficient tends to zero, the model formally tends to a mixed formulation of Poisson's equation with homogeneous Neumann boundary conditions. When viscosity coefficient equals to zero the model has the form of Darcy's law for flow in a homogeneous porous medium, with a volume averaged velocity. In fact, the Brinkman model can be regarded as a macroscopic model for flow in an "almost porous media". However, many researches have focused on Stokes problem or Darcy problem. Several successful finite elements to solve Stokes problem have been proposed and used. For instance conforming finite element spaces [4, 16, 44, 46] including the $([P_2]^2, P_0)$ and $([P_2]^2, P_1)$ (the Taylor-Hood element) elements [22, 27] and the MINI element [2]. Compared with conforming finite element space which allows the continuity throughout the interfaces, nonconforming finite elements [13, 25, 42, 15, 10] have been developed so that the discrete inf-sup condition is satisfied. The basis functions of nonconforming finite elements have simple and small support sets, which allow much more efficient parallel implementations. Moreover, they allow a natural coupling with $H(\mathrm{div}\,)$ element which is advantageous for porous media. Some researchers want to treat both porous media flow and open fluid flow, so it would be advantageous if the same element could be used in both the Stokes limit and Darcy limit [14, 19, 33]. The promising candidate for such an element is the nonconforming Crouzeix-Raviart element [13, 43] which has several useful properties. However, Mardal, Tai, and Winther [37] showed that the CR element does not converge. By adding an edge stabilization term, Burman and Hansbo [8] proved that the simplest $([P_1]^2, P_0)$ element can be used for both Darcy and Stokes problems, but the choice of stabilization parameter requires special care. Also various finite element families for the Stokes-Darcy problem are tested numerically in [26]. In 2008, Xie, Xu and Xue [48] developed a class of low order simplex elements for both two and three dimensions which are uniform, stable with respect to the viscosity coefficient, zero-order term coefficient, and their jumps. In the other direction, Karper, Mardal, and Winther [31] discussed how the coupled problem can be solved by using standard Stokes elements like the MINI element or the Taylor-Hood element in the entire domain.

In 2003, Park and Sheen [39] presented a P_1–nonconforming quadrilateral element to solve second-order elliptic problems which consists of linear polynomials only on each quadrilateral and the barycenter values DOF at the element interfaces. Recently, Kim, Yim and Sheen [32] proved that this finite element and piecewise constant pair by removing the global one-dimensional checkerboard pattern satisfies discrete inf-sup condition. Another direction is to add a globally one-dimensional $DSSY$-type (or Rannacher-Turek type) bubble space based on macro interior edges, with the simple piecewise constant pressure element unmodified. Our aim is to apply these finite element pair to Stokes-Darcy-Brinkman interface problem. Since P_1–nonconforming can not solve the Darcy problem, we will use Raviart-Thomas element to approximate velocity field. In particular, the mathematical theory and the associated numerical analysis of a mixed variational formulation were provided in [33]. There, the coupling across the interface is determined by the Beavers-Joseph-Saffman conditions, which yield the introduction of the trace of the porous medium pressure as a suitable Lagrange multiplier. In addition, well posedness of the corresponding continuous formulation and a detailed analysis of a nonconforming mixed finite element method are given in [33]. In a similar direction, [20] used Bernardi-Raugel [4, 21] and Raviart-Thomas elements for the velocites, piecewise constants for the pressures and continuous piecewise linear elements for the Lagrange multiplier, then show stability, convergence and *a priori* error estimates for the associated Galerkin scheme. On the other hand, Huang and Chen [30] used to Crouzeix-Raviart element or Rannacher-Turek element to approximate one component of the velocity and the other component is approximated by conforming P_1 or Q_1 element in Stokes equations while Raviart-Thomas element is used on Darcy problem. Arbogast and Brunson [1] modified certain conforming Stokes element slightly near the interface to account for the tangential discontinuity. This gives a mixed finite element method for the entire Stokes-Darcy system.

The outline of this paper is organized as follows. In Section 2, the model problem will be stated and its mixed variational formulation will be defined. In Section 3, we introduce the nonconforming mixed finite element method. For Stokes-Brinkman problem, we will use P_1–nonconforming elements for the velocity in the entire domain, while for Stokes-Darcy problem, P_1–nonconforming elements will be used to approximate the velocity in fluid region, lowest order Raviart-Thomas element for the velocity in porous medium, piecewise constants for the pressures. In Section 4, we will devote to check the discrete inf-sup condition for our proposed finite element pair. In Section 5, we construct the interpolation operator and establish the corresponding *a priori* error estimate. Finally, some numerical experiments for the two coupled problems are shown in Section 6.

2.2 The Stokes-Brinkman and Stokes-Darcy interface models

2.2.1 The models

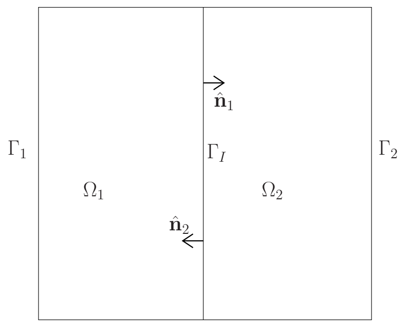


Figure 2.1: The domain for model problem

Let Ω be a domain in $\mathbb{R}^2$, composed by two subdomains Ω_1 and Ω_2. The model we consider consists of the fluid region Ω_1 and the porous medium domain Ω_2. These are separated by an interface Γ_I. Here the subdomain Ω_1 and Ω_2 are assumed to be bounded connected polygonal domains such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ with outward unit normal vectors $\widehat{\mathbf{n}}_j,\ j = 1, 2$. Let $\Gamma_j := \partial\Omega_j \backslash \Gamma_I$. Figure 2.1 gives a schematic representation of the geometry. Denote by $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ the velocity and by $p = (p_1, p_2)$ the fluid pressure so that $\mathbf{u}_j = \mathbf{u}|_{\Omega_j}$ and $p_j = p|_{\Omega_j},\ j = 1, 2$. The flow in the domain Ω_1 is assumed to be of Stokes type. Hence, $(\mathbf{u}_1, p_1)$ satisfies

$$
\begin{aligned}
-\nu_1 \Delta \mathbf{u}_1 + \nabla p_1 &= \mathbf{f_1} \quad \text{in } \Omega_1, && (2.1a)\\
\nabla \cdot \mathbf{u}_1 &= 0 \quad \text{in } \Omega_1, && (2.1b)\\
\mathbf{u}_1 &= \mathbf{0} \quad \text{on } \Gamma_1, && (2.1c)
\end{aligned}
$$

where $\nu_1 > 0$ the viscosity and vector field $\mathbf{f_1}$ is the body force. Assuming porous medium domain Ω_2, $(\mathbf{u}_2, p_2)$ satisfies

$$
\begin{aligned}
-\nu_2 \Delta \mathbf{u}_2 + \mathbf{u}_2 + \nabla p_2 &= \mathbf{f_2} \quad \text{in } \Omega_2, && (2.2a)\\
\nabla \cdot \mathbf{u}_2 &= g \quad \text{in } \Omega_2, && (2.2b)\\
\mathbf{u}_2 &= \mathbf{0} \quad \text{on } \Gamma_2, && (2.2c)
\end{aligned}
$$

where $\nu_2 \in [0, 1]$ and vector field $\mathbf{f_2}$ is the body force. The source g is assumed to satisfy the solvability condition

$$\int_{\Omega_2} g \; d\mathbf{x} = 0,$$

which makes physical sence due to the no-flow boundary condition on $\partial\Omega$ and interface condition below. We note that when ν_2 is not too small, and $g = 0$, the problem is reduced to standard Stokes problem but with an additional zero order term. However, when ν_2 approaches zero, then the model problem (2.2) is close to the Darcy problem. When $\nu_2 = 0$, the first equation in (2.2) has the form of Darcy's law for flow in a homogeneous porous medium.

Remark 2.2.1. *When we refer to the Darcy equation corresponding to* (2.2)*, we refer to the system* (2.2) *with* $\nu_2 = 0$ *and the boundary condition* $\mathbf{u}_2 = 0$ *replaced by* $\mathbf{u}_2 \cdot \widehat{\mathbf{n}} = 0$*.*

2.2.2 Interface conditions

The problems (2.1)-(2.2) must be coupled across Γ_I by the correct interface conditions.

1. For the Stokes-Brinkman problem, two sets of interface conditions are widely used. Standard continuity of the velocity is expressed by

$$\mathbf{u}_1 = \mathbf{u}_2 \qquad \text{on} \ \ \Gamma_I. \tag{2.3}$$

   The second interface condition is the continuity of normal forces given by

$$(\nu_1 \nabla \mathbf{u}_1 - p_1 \mathbf{I}) \cdot \widehat{\mathbf{n}}_1 = (\nu_2 \nabla \mathbf{u}_2 - p_2 \mathbf{I}) \cdot \widehat{\mathbf{n}}_1 \qquad \text{on} \ \ \Gamma_I. \tag{2.4}$$

2. For the Stokes-Darcy problem, three sets of interface conditions are also widely used. Mass conservation across Γ_I is expressed by

$$\mathbf{u}_1 \cdot \widehat{\mathbf{n}}_1 = \mathbf{u}_2 \cdot \widehat{\mathbf{n}}_1 \qquad \text{on} \ \ \Gamma_I. \tag{2.5}$$

   The second interface condition is the continuity of normal forces given by

$$\nu_1 \widehat{\mathbf{n}}_1 \cdot \nabla \mathbf{u}_1 \cdot \widehat{\mathbf{n}}_1 = p_1 - p_2 \qquad \text{on} \;\; \Gamma_I. \tag{2.6}$$

Finally, since the fluid model is viscous, a condition on the tangential fluid velocity on Γ_I must be given. We will consider the third interface condition as Beavers-Joseph-Saffmann condition.

$$\widehat{\tau}_1 \cdot \nabla \mathbf{u}_1 \cdot \widehat{\mathbf{n}}_1 = -\alpha \mathbf{u}_1 \cdot \widehat{\tau}_1 \qquad \text{on} \;\; \Gamma_I, \tag{2.7}$$

where $\alpha > 0$ is the friction constant determined experimentally and the Beavers-Joseph-Saffman law establishes that the slip velocity along Γ_I is proportional to the shear stress along Γ_I. For more details, we refer to [3, 45].

2.2.3 Weak formulation of the two coupled models

This subsection is devoted to developing suitable weak formulations of the problems (2.1)-(2.2). First, we will introduce a weak formulation for the Stokes-Brinkman problem with the interface condition (2.3)-(2.4). Set

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \mid \int_\Omega q \; d\mathbf{x} = 0 \right\}.$$

Here, and in what follows, we use the standard notations and definitions for the Sobolev spaces $[H^s(S)]^2$, and their associated inner products $(\cdot,\cdot)_{s,S}$, norms $||\cdot||_{s,S}$, and semi-norms $|\cdot|_{s,S}$. We will omit the subscripts s, S if $s = 0$ and $S = \Omega$. Also for boundary ∂S of S, the inner product in $L^2(\partial S)$ is denoted by $\langle \cdot,\cdot \rangle_S$. To derive the weak formulation, let $\mathbf{v}_j := \mathbf{v}|_{\Omega_j}$ and $q_j := q|_{\Omega_j}, \; j = 1, 2$ where $\mathbf{v} \in [H_0^1(\Omega)]^2$ and $q \in L_0^2(\Omega)$, respectively. multiplying by a function $\mathbf{v}_1$

in (2.1a), and intergrating by parts gives

$$(\mathbf{f}_1, \mathbf{v}_1)_{\Omega_1} = \nu_1(\nabla \mathbf{u}_1, \nabla \mathbf{v}_1)_{\Omega_1} - \nu_1 \langle \nabla \mathbf{u}_1 \cdot \widehat{\mathbf{n}}_1, \mathbf{v}_1 \rangle_{\Gamma_I}$$
$$-(\nabla \cdot \mathbf{v}_1, p_1)_{\Omega_1} + \langle \mathbf{v}_1 \cdot \widehat{\mathbf{n}}_1, p_1 \rangle_{\Gamma_I}.$$

Therefore, introducing the bilinear forms

$$a_1(\mathbf{u}_1, \mathbf{v}_1) := \nu_1(\nabla \mathbf{u}_1, \nabla \mathbf{v}_1)_{\Omega_1}, \qquad b_1(\mathbf{v}_1, q_1) := (\nabla \cdot \mathbf{v}_1, q_1)_{\Omega_1},$$

we obtain for all $\mathbf{v}_1 := \mathbf{v}|_{\Omega_1}$ and $q_1 := q|_{\Omega_1}$,

$$\begin{aligned} a_1(\mathbf{u}_1, \mathbf{v}_1) - b_1(\mathbf{v}_1, p_1) - \langle (\nu_1 \nabla \mathbf{u}_1 - p_1 \mathbf{I}) \cdot \widehat{\mathbf{n}}_1, \mathbf{v}_1 \rangle_{\Gamma_I} &= (\mathbf{f}_1, \mathbf{v}_1)_{\Omega_1}, \\ b_1(\mathbf{u}_1, q_1) &= 0. \end{aligned}$$

In the porous medium domain, multiplication of the first equation (2.2a) by $\mathbf{v}_2$, integration over Ω_2, and integration by parts gives

$$(\mathbf{f}_2, \mathbf{v}_2)_{\Omega_2} = \nu_2(\nabla \mathbf{u}_2, \nabla \mathbf{v}_2)_{\Omega_2} - \nu_2 \langle \nabla \mathbf{u}_2 \cdot \widehat{\mathbf{n}}_2, \mathbf{v}_2 \rangle_{\Gamma_I} + (\mathbf{u}_2, \mathbf{v}_2)_{\Omega_2}$$
$$-(\nabla \cdot \mathbf{v}_2, p_2)_{\Omega_2} + \langle \mathbf{v} \cdot \widehat{\mathbf{n}}_2, p_2 \rangle_{\Gamma_I},$$

Introducing

$$a_2(\mathbf{u}_2, \mathbf{v}_2) := \nu_2(\nabla \mathbf{u}_2, \nabla \mathbf{v}_2)_{\Omega_2} + (\mathbf{u}_2, \mathbf{v}_2)_{\Omega_2}, \qquad b_2(\mathbf{v}_2, q_2) := (\nabla \cdot \mathbf{v}_2, q_2)_{\Omega_2},$$

we have

$$\begin{aligned} a_2(\mathbf{u}_2, \mathbf{v}_2) - b_2(\mathbf{v}_2, p_2) - \langle (\nu_2 \nabla \mathbf{u}_2 - p_2 \mathbf{I}) \cdot \widehat{\mathbf{n}}_2, \mathbf{v}_2 \rangle_{\Gamma_I} &= (\mathbf{f}_2, \mathbf{v}_2)_{\Omega_2}, \\ b_2(\mathbf{u}_2, q_2) &= (g, q_2)_{\Omega_2}. \end{aligned}$$

By interface condition (2.3)-(2.4) for Stokes-Brinkman problem, we have

$$
\begin{aligned}
a(\mathbf{u},\mathbf{v}) &:= \sum_{i=1}^{2} a_i(\mathbf{u}_i,\mathbf{v}_i) : [H_0^1(\Omega)]^2 \times [H_0^1(\Omega)]^2 \to \mathbb{R},\\
b(\mathbf{v},p) &:= \sum_{i=1}^{2} b_i(\mathbf{v}_i,p_i) : [H_0^1(\Omega)]^2 \times L_0^2(\Omega) \to \mathbb{R},\\
l(\mathbf{v}) &:= \sum_{i=1}^{2} (\mathbf{f}_i,\mathbf{v}_i)_{\Omega_i}, \quad g(q) := (g,q_2)_{\Omega_2}.
\end{aligned}
$$

Then, the weak formulation of (2.1)-(2.2) is to seek a pair $(\mathbf{u},p) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ such that

$$
\begin{aligned}
a(\mathbf{u},\mathbf{v}) - b(\mathbf{v},p) &= l(\mathbf{v}) \quad \forall \mathbf{v} \in [H_0^1(\Omega)]^2, &\text{(2.8a)}\\
b(\mathbf{u},q) &= g(q) \quad \forall q \in L_0^2(\Omega), &\text{(2.8b)}
\end{aligned}
$$

Next, we will derive the weak formulation for the Stokes-Darcy problem. In fact, for the reduced problem (2.1)-(2.2) when $\nu_2 = 0$, the space $[H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ in not a proper function space for the solution. Hence, let

$$
\mathbf{X}_1 := \{\mathbf{v}_1 \in [H^1(\Omega_1)]^2 : \mathbf{v}_1 = 0 \text{ on } \Gamma_1\}, \qquad M_1 := L^2(\Omega_1)
$$

denote the usual velocity-pressure spaces on Ω_1. The norm on $\mathbf{X}_1$ is given by

$$
\|\mathbf{v}_1\|_{\mathbf{X}_1} := |\mathbf{v}_1|_{1,\Omega_1}.
$$

The velocity space X_2 on Ω_2 is the subspace of

$$
H(\mathrm{div};\Omega_2) = \{\mathbf{v}_2 \in [L^2(\Omega_2)]^2 : \nabla \cdot \mathbf{v}_2 \in L^2(\Omega_2)\}
$$

consisting of functions with zero normal trace on Γ_2 and equipped with the norm

$$\|\mathbf{v}_2\|_{H(\text{div};\Omega_2)} := (\|\mathbf{v}_2\|^2_{\Omega_2} + \|\nabla \cdot \mathbf{v}_2\|^2_{\Omega_2})^{1/2}.$$

It is well known that for all $\mathbf{v}_2 \in H(\text{div};\Omega_2)$, $\mathbf{v}_2 \cdot \widehat{\mathbf{n}}_2 \in H^{-1/2}(\partial\Omega_2)$. However, The restriction of $\mathbf{v}_2 \cdot \widehat{\mathbf{n}}_2$ to Γ_2 may not lie in $H^{-1/2}(\Gamma_2)$. Thus, we define the velocity-pressure spaces on Ω_2 as follows:

$$\mathbf{X}_2 := \{\mathbf{v}_2 \in H(\text{div};\Omega_2) : \langle \mathbf{v}_2 \cdot \widehat{\mathbf{n}}_2, w\rangle_{\partial\Omega_2} = 0 \quad \forall w \in H^1_{\Gamma_I}(\Omega_2)\}, \qquad M_2 := L^2(\Omega_2),$$

where

$$H^1_{\Gamma_I}(\Omega_2) = \{w \in H^1(\Omega_2) : w = 0 \text{ on } \Gamma_I\}.$$

Defining $\mathbf{X} := \mathbf{X}_1 \times \mathbf{X}_2$ which $\mathbf{v} \in \mathbf{X}$ takes the form $(\mathbf{v}_1, \mathbf{v}_2)$ with $\mathbf{v}_i \in \mathbf{X}_i$. The norm on $\mathbf{X}$ is given by

$$\|\mathbf{v}\|_{\mathbf{X}} := (\|\mathbf{v}_1\|^2_{\mathbf{X}_1} + \|\mathbf{v}_2\|^2_{\mathbf{X}_2})^{1/2} \qquad \mathbf{v} \in X.$$

Similarly, let

$$M := \{q = (q_1, q_2) : q_i \in M_i \text{ and } \sum_{i=1}^{2} (q_i, 1)_{\Omega_i} = 0\},$$

with norm

$$\|q\|_M := (\|q_1\|^2_{M_1} + \|q_2\|^2_{M_2})^{1/2}.$$

In order to handle interface Γ_I between the subdomains in Ω_1 and Ω_2, we will introduce the Lagrange multiplier λ in (2.6) as follows:

$$p_1 - \nu_1 \widehat{\mathbf{n}}_1 \cdot \nabla \mathbf{u}_1 \cdot \widehat{\mathbf{n}}_1 = \lambda = p_2 \qquad \text{on } \Gamma_I. \tag{2.9}$$

Hence, we obtain weak formulation of (2.1) as follows. For a sufficiently smooth $\mathbf{v}_1 \in \mathbf{X}_1$, multiplying and integrating by parts gives

$$\begin{aligned}
(\mathbf{f}_1, \mathbf{v}_1)_{\Omega_1} &= \nu_1(\nabla \mathbf{u}_1, \nabla \mathbf{v}_1)_{\Omega_1} - \nu_1 \langle \nabla \mathbf{u}_1 \cdot \widehat{\mathbf{n}}_1, \mathbf{v}_1 \rangle_{\Gamma_I} - (\nabla \cdot \mathbf{v}_1, p_1)_{\Omega_1} + \langle \mathbf{v}_1 \cdot \widehat{\mathbf{n}}_1, p_1 \rangle_{\Gamma_I}, \\
&= \nu_1(\nabla \mathbf{u}_1, \nabla \mathbf{v}_1)_{\Omega_1} - (\nabla \cdot \mathbf{v}_1, p_1)_{\Omega_1} \\
&+ \langle p_1 - \nu_1 \widehat{\mathbf{n}}_1 \cdot \nabla \mathbf{u}_1 \cdot \widehat{\mathbf{n}}_1, \mathbf{v}_1 \cdot \widehat{\mathbf{n}}_1 \rangle_{\Gamma_I} - \nu_1 \langle \widehat{\tau}_1 \cdot \nabla \mathbf{u}_1 \cdot \widehat{\mathbf{n}}_1, \mathbf{v}_1 \cdot \widehat{\tau}_1 \rangle_{\Gamma_I},
\end{aligned}$$

By interface condition (2.7) and (2.9), we obtain

$$(\mathbf{f}_1, \mathbf{v}_1)_{\Omega_1} = \nu_1(\nabla \mathbf{u}_1, \nabla \mathbf{v}_1)_{\Omega_1} - (\nabla \cdot \mathbf{v}_1, p_1)_{\Omega_1} + \langle \lambda, \mathbf{v}_1 \cdot \widehat{\mathbf{n}}_1 \rangle_{\Gamma_I} + \nu_1 \alpha \langle \mathbf{u}_1 \cdot \widehat{\tau}_1, \mathbf{v}_1 \cdot \widehat{\tau}_1 \rangle_{\Gamma_I}.$$

Therefore, introducing the bilinear forms

$$a_1(\mathbf{u}_1, \mathbf{v}_1) := \nu_1(\nabla \mathbf{u}_1, \nabla \mathbf{v}_1)_{\Omega_1} + \nu_1 \alpha \langle \mathbf{u}_1 \cdot \widehat{\tau}_1, \mathbf{v} \cdot \widehat{\tau}_1 \rangle_{\Gamma_I}, \qquad b_1(\mathbf{v}, q_1) := (\nabla \cdot \mathbf{v}_1, q_1)_{\Omega_1},$$

we obtain for all $\mathbf{v}_1 \in \mathbf{X}_1$ and $q_1 \in M_1$

$$\begin{aligned}
a_1(\mathbf{u}_1, \mathbf{v}_1) - b_1(\mathbf{v}_1, p_1) + \langle \lambda, \mathbf{v}_1 \cdot \widehat{\mathbf{n}}_1 \rangle_{\Gamma_I} &= (\mathbf{f}_1, \mathbf{v})_{\Omega_1}, \\
b_1(\mathbf{u}_1, q_1) &= 0.
\end{aligned}$$

In the porous medium domain, multiplication of the first equation (2.2) by $\mathbf{v}_2 \in \mathbf{X}_2$, integration over Ω_2, and integration by parts gives

$$(\mathbf{f}_2, \mathbf{v})_{\Omega_2} = (\mathbf{u}_2, \mathbf{v}_2)_{\Omega_2} - (\nabla \cdot \mathbf{v}_2, p_2)_{\Omega_2} + \langle \lambda, \mathbf{v} \cdot \widehat{\mathbf{n}}_2 \rangle_{\Gamma_I},$$

where, by (2.9), p_2 is replaced by λ in the last term. Introducing

$$a_2(\mathbf{u}_2, \mathbf{v}_2) := (\mathbf{u}_2, \mathbf{v}_2)_{\Omega_2}, \qquad b_2(\mathbf{v}_2, q_2) := (\nabla \cdot \mathbf{v}_2, q_2)_{\Omega_2},$$

we have for all $\mathbf{v}_2 \in \mathbf{X}_2$ and $q_2 \in M_2$

$$\begin{aligned} a_2(\mathbf{u}_2, \mathbf{v}_2) - b_2(\mathbf{v}_2, p_2) + \langle \lambda, \mathbf{v}_2 \cdot \widehat{\mathbf{n}}_2 \rangle_{\Gamma_I} &= (\mathbf{f}_2, \mathbf{v}_2)_{\Omega_2}, \\ b_2(\mathbf{u}_2, q_2) &= (g, q_2)_{\Omega_2}. \end{aligned}$$

In the reference [33], authors mentioned that the linking across Γ_I occurs through the condition $\mathbf{u}_1 \cdot \widehat{\mathbf{n}}_1 + \mathbf{u}_2 \cdot \widehat{\mathbf{n}}_2 = 0$ on Γ_I and the definition (2.9) of λ. This is the key to choose Λ for the Lagrange multipliers for well-posedness of the coupled problem for Stokes-Darcy problem. Define

$$\Lambda := H_{00}^{1/2}(\Gamma_I)(\subset L^2(\Gamma_I)),$$

where $H_{00}^{1/2}(\Gamma_I)$ is the completion of the smooth functions with compact support in Γ_I. We note that any function $\mu \in H_{00}^{1/2}(\Gamma_I)$ has the property which its extension by zero to $\partial\Omega_j$ gives a function $\tilde{\mu}_j \in H^{1/2}(\partial\Omega_j)$ with

$$\|\tilde{\mu}_j\|_{1/2,\partial\Omega_j} \leq C\|\mu\|_{H_{00}^{1/2}(\Gamma_I)}, \quad j = 1, 2. \tag{2.10}$$

Hence, we define

$$b_I(\mathbf{v}, \lambda) := \langle \mathbf{v}_1 \cdot \widehat{\mathbf{n}}_1 + \mathbf{v}_2 \cdot \widehat{\mathbf{n}}_2, \lambda \rangle : \mathbf{X} \times \Lambda \to \mathbb{R}.$$

The flux continuity condition on Γ_I implies

$$b_I(\mathbf{v}, \lambda) = 0 \qquad \forall \lambda \in \Lambda.$$

Then, the weak formulation of (2.1)-(2.2) is to seek a pair $(\mathbf{u}, p, \lambda) \in \mathbf{X} \times M \times \Lambda$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + b(\mathbf{v}, \lambda) &= l(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{X}, \\ b(\mathbf{u}, q) &= g(q) \quad \forall q \in M, \\ b_I(\mathbf{u}, \mu) &= 0 \qquad \forall \mu \in \Lambda, \end{aligned}$$

where

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &:= \sum_{i=1}^{2} a_i(\mathbf{u}_i, \mathbf{v}_i) : \mathbf{X} \times \mathbf{X} \to \mathbb{R}, \\ b(\mathbf{v}, p) &:= \sum_{i=1}^{2} b_i(\mathbf{v}_i, p_i) : \mathbf{X} \times M \to \mathbb{R}, \\ l(\mathbf{v}) &:= \sum_{i=1}^{2} (\mathbf{f}_i, \mathbf{v}_i)_{\Omega_i}, \quad g(q) := (g, q_2)_{\Omega_2}. \end{aligned}$$

Another weak formulation can be derived by using the space $\mathbf{V}$ of functions in $\mathbf{X}$ with trace-continuous normal velocities:

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{X} : b_I(\mathbf{v}, \lambda) = 0 \quad \forall \mu \in \Lambda\}.$$

Several properties for the space V have been investigated in [33]. Thus, we can write another weak formulation as follows: find a pair $(\mathbf{u}, p) \in \mathbf{V} \times M$ such that

$$a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) = l(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}, \tag{2.11a}$$

$$b(\mathbf{u}, q) = g(q) \quad \forall q \in M. \tag{2.11b}$$

2.3 Finite element discretization

In this section, we will consider the finite element discretization of the two coupled problems. Throughout the paper, we shall restrict our attention to the case of $\Omega = [0,1]^2$. Let $(\mathcal{T}_{h,j})_{0<h<1}$ be a family of uniform triangulation of $\Omega_j,\ j = 1,2$ into disjoint squares Q of size h with $\overline{\Omega}_j = \bigcup_{Q\in\mathcal{T}_{h,j}} \overline{Q}$. For the standard definition of regular partition, we refer to [12, 21]. $\mathcal{T}_h$ denotes a family of uniform triangulation of Ω such that $\mathcal{T}_h = \cup_{j=1}^{2} \mathcal{T}_{h,j}$ and assume that the meshes $\mathcal{T}_{h,1}$ and $\mathcal{T}_{h,2}$ match at Γ_I. Let N_Q and $N_{Q,j}$ denote the number of elements in Ω and $\Omega_j,\ j = 1,2$, respectively. We use the following notation.

$$\mathcal{E}_h(\Gamma_I) \quad := \quad \text{the set of all element edges } E \text{ with } E \subset \Gamma_I.$$

First, we will investigate the case of Stokes-Brinkman problem. The approximate space is based on the P_1–nonconforming quadrilateral element [39] for velocity fields and piecewise constant for pressure fields. Set

$$\begin{aligned}\mathcal{NC}_0^h = \{v \in L^2(\Omega) \mid\ v|_Q \in P_1(Q)\ \forall Q \in \mathcal{T}_h, v \text{ is continuous at the midpoint}\\ \text{of each interior edge in } \mathcal{T}_h,\ \text{and } v \text{ vanishes at the midpoint}\\ \text{of each boundary edge in } \mathcal{T}_h\}.\end{aligned}$$

The pressure will be approximated by the piecewise constant space as follows:

$$P^h = \{q \in L_0^2(\Omega) \mid q|_Q \in P_0(Q)\ \forall Q \in \mathcal{T}_h\}.$$

It is well known that the pair of spaces $([\mathcal{NC}_0^h]^2, P^h)$ cannot be used to solve the problem since Theorem 3.2 in [38]. Hence our suitable finite element space the pair to solve the problem is $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$, where $\mathcal{P}_{cf}^h$ is removed a global checkerboard pattern from P^h. (For more details, see [32]). Another way is to add a bubble function to the P_1–nonconforming element by using another quadrilateral nonconforming bubble function [15, 42]. In this paper, we focus on the pair of spaces $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$. Now define the discrete weak formulation of Stokes-Brinkman problem to find a pair $(\mathbf{u}_h, p_h) \in [\mathcal{NC}_0^h]^2 \times \mathcal{P}_{cf}^h$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) - b_h(\mathbf{v}_h, p_h) = l(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in [\mathcal{NC}_0^h]^2, \tag{2.12a}$$

$$b_h(\mathbf{u}_h, q_h) = g(q_h) \quad \forall q_h \in \mathcal{P}_{cf}^h, \tag{2.12b}$$

where the discrete bilinear forms $a_h(\cdot,\cdot) : [\mathcal{NC}_0^h]^2 \times [\mathcal{NC}_0^h]^2 \to \mathbb{R}$ and $b_h(\cdot,\cdot) : [\mathcal{NC}_0^h]^2 \times \mathcal{P}_{cf}^h \to \mathbb{R}$ are defined in the standard fashion:

$$\begin{aligned} a_h(\mathbf{u}, \mathbf{v}) &= \nu_1 \sum_{j=1}^{N_{Q,1}} (\nabla \mathbf{u}, \nabla \mathbf{v})_{Q_j} + \sum_{j=1}^{N_{Q,2}} \nu_2 (\nabla \mathbf{u}, \nabla \mathbf{v})_{Q_j} + (\mathbf{u}, \mathbf{v})_{Q_j} \\ b_h(\mathbf{v}, q) &= \sum_{j=1}^{N_Q} (\nabla \cdot \mathbf{v}, q)_{Q_j}. \end{aligned}$$

As usual, let $|||\cdot|||_h$ denote the (broken) energy norm given by

$$|||\mathbf{v}|||_h = \sqrt{a_h(\mathbf{v}, \mathbf{v})}.$$

Second, we will investigate the case of Stokes-Darcy problem. For the discretization of fluid's region Ω_1, we still use P_1–nonconforming finite element for velocity defined by

$$\begin{aligned} \mathcal{NC}_{\Gamma_1}^h = \{ v \in L^2(\Omega_1) \mid v|_Q \in P_1(Q) \ \forall Q \in \mathcal{T}_{h,1}, & \ v \text{ is continuous at the midpoint} \\ & \text{of each interior edge in } \mathcal{T}_{h,1}, \text{ and } v \text{ vanishes at the midpoint} \\ & \text{of each edge on } \Gamma_1 \}. \end{aligned}$$

In the porous medium domain Ω_2, we choose $\mathbf{X}_2^h \subset \mathbf{X}^h$ to be the well known mixed finite element spaces, the lowest order RT spaces [43]. In the rest of the paper, let $\mathbf{X}_1^h := [\mathcal{NC}_{\Gamma_1}^h]^2$ and C^h be a global checkerboard pattern function on Ω_1. Finally, we define the finite element spaces over Ω:

$$\mathbf{X}^h := \mathbf{X}_1^h \times \mathbf{X}_2^h, \qquad M^h := \left\{ (q_1, q_2) \in M_{1,cf}^h \times M_2^h : \int_{\Omega_1} q_1 \; d\mathbf{x} + \int_{\Omega_2} q_2 \; d\mathbf{x} = 0 \right\},$$

where

$$\begin{aligned} M_{1,cf}^h &= \left\{ q_h \in L^2(\Omega_1) \;\middle|\; q_h|_Q \in P_0(Q), \quad \forall Q \in \mathcal{T}_{h,1} \;\text{ and }\; (q_h, C^h)_{\Omega_1} = 0 \right\}, \\ M_2^h &= \left\{ q_h \in L^2(\Omega_2) \;\middle|\; q_h|_Q \in P_0(Q), \quad \forall Q \in \mathcal{T}_{h,2} \right\} \end{aligned}$$

Similar to continuous level, we define the discrete norm for all $\mathbf{v}_h = (\mathbf{v}_{h,1}, \mathbf{v}_{h,2}) \in \mathbf{X}^h$ as follows:

$$\|\mathbf{v}_h\|_{\mathbf{X}^h} = \left(|\mathbf{v}_{h,1}|_{1,h,\Omega_1}^2 + \|\mathbf{v}_{h,2}\|_{H(\mathrm{div}\,;\Omega_2)}^2 \right)^{1/2},$$

where

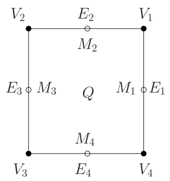
$$|\mathbf{v}|_{1,h,\Omega_1} = \left[\sum_{Q \in \mathcal{T}_{h,1}} |\mathbf{v}|_{1,Q}^2 \right]^{1/2}.$$

Also, let

$$\Lambda^h := \{ \mu_h \in L^2(\Gamma_I) : \mu_h|_E \in P_0(E) \quad \forall E \in \mathcal{E}_h(\Gamma_I) \}.$$

With this Λ^h, we define

$$\mathbf{V}^h := \{ \mathbf{v}_h = (\mathbf{v}_{h,1}, \mathbf{v}_{h,2}) \in \mathbf{X}^h : b_I(\mathbf{v}_h, \mu_h) = 0 \quad \forall \mu_h \in \Lambda^h \}.$$



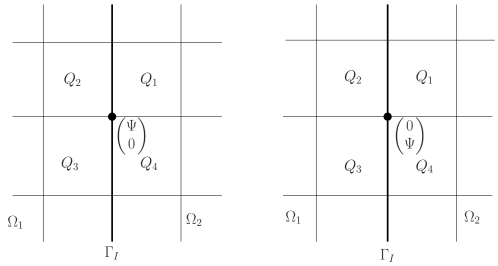


Figure 2.2: (Top) A square with vertices V_j's, edges E_j's, and midpoints M_j's. (Bottom) The two basis functions defined on the interface Γ_I.

Remark 2.3.1. *The space Λ^h is the normal trace of $\mathbf{X}_2^h$ on Γ_I.*

Now define the discrete weak formulation of Stokes-Darcy problem to find a pair $(\mathbf{u}_h, p_h) \in \mathbf{V}^h \times M^h$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) - b_h(\mathbf{v}_h, p_h) = l(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}^h, \tag{2.13a}$$

$$b_h(\mathbf{u}_h, q_h) = g(q_h) \quad \forall q_h \in M^h, \tag{2.13b}$$

where the discrete bilinear forms $a_h(\cdot,\cdot) : \mathbf{V}^h \times \mathbf{V}^h \to \mathbb{R}$ and $b_h(\cdot,\cdot) : \mathbf{V}^h \times M^h \to \mathbb{R}$ are defined in the standard fashion:

$$\begin{aligned} a_h(\mathbf{u}, \mathbf{v}) &= \nu_1 \sum_{j=1}^{N_{Q,1}} (\nabla \mathbf{u}, \nabla \mathbf{v})_{Q_j} + \nu_1 \alpha \sum_{E \in \mathcal{E}_h(\Gamma_I)} \langle \mathbf{u} \cdot \widehat{\tau}_1, \mathbf{v} \cdot \widehat{\tau}_1 \rangle_E + \sum_{j=1}^{N_{Q,2}} (\mathbf{u}, \mathbf{v})_{Q_j} \\ b_h(\mathbf{v}, q) &= \sum_{j=1}^{N_Q} (\nabla \cdot \mathbf{v}, q)_{Q_j}. \end{aligned}$$

2.3.1 A closer look at the space $\mathbf{V}^h$

In order to solve the discrete reduced weak formulation (2.13), it is important to understand in exactly what sense mass conservation across Γ_I (2.5) holds. By definition of $\mathbf{V}^h$, conservation of mass across Γ_I holds only in an approximation sense. To this end, a local characterization of the functions $\mathbf{v} \in \mathbf{V}^h$ is needed. If a function $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{X}^h$ belongs to $\mathbf{V}^h$, then the nodal values of $\mathbf{v}_2 \cdot \widehat{\mathbf{n}}_2 \in \mathbf{X}_2^h$ on Γ_I are linked to those of $\mathbf{v}_1 \cdot \widehat{\mathbf{n}}_1$ on Γ_I. Let $\mathcal{F}_j$ denote the set of nodes of $\mathbf{X}_j^h$, $j = 1, 2$, and $\mathcal{F}_j(E)$ the set of nodes $k \in \mathcal{F}_j$ belonging to an element edge E, and let ϕ_k^j, $k \in \mathcal{F}_j$ ($j = 1, 2$), be the associated basis functions of $\mathbf{X}_j^h$. By the following proposition, we have the characterization of the space $\mathbf{v} \in \mathbf{V}^h$.

Proposition 2.3.2. *[33] Let $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{X}^h$ be given. Then $\mathbf{v} \in \mathbf{V}^h$ is equivalent to the following relation between the nodal values $v_j^{(1)}$ and $v_j^{(2)}$ of $\mathbf{v}_1$ and $\mathbf{v}_2$ on E being satisfied*

$$v_j^{(2)} = -|E|^{-1} \sum_{k \in \mathcal{F}_1(E)} v_k^{(1)} \left\langle \phi_k^{(1)} \cdot \widehat{\mathbf{n}}_1, 1 \right\rangle_E \qquad \textit{for all } j \in \mathcal{F}_2(E), \quad E \in \mathcal{E}_h(\Gamma_I).$$

For our proposed nonconforming finite element, we get the following basis functions on $E \in \mathcal{E}_h(\Gamma_I)$. Let $\varphi_j : Q \to \mathbb{R}$ and $\psi_j : Q \to \mathbb{R}$, $j = 1, \cdots, 4$ denote the P_1–nonconforming and RT local basis functions (see Figure 2.2), respectively, defined by

$$\varphi_j(m) = \begin{cases} 1, & \text{if } m = M_j, M_{j+1} \\ 0, & \text{if } m = M_{j+2}, M_{j+3}, \end{cases} \qquad \psi_j|_{E_k} \cdot \nu_k = \begin{cases} 1, & \text{if } j = k \\ 0, & \text{Otherwise,} \end{cases}$$

where ν_k be a outward unit normal vectors on the edges E_k on Q. Then the basis functions $\begin{pmatrix} \Psi \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \Psi \end{pmatrix}$ as depicted in Figure 2.2 can be understood by

$$\begin{pmatrix}\Psi\\0\end{pmatrix}=\begin{cases}\begin{pmatrix}-\psi_3\\0\end{pmatrix} & \text{on} \quad Q_1\\ \begin{pmatrix}\varphi_3\\0\end{pmatrix} & \text{on} \quad Q_2\\ \begin{pmatrix}\varphi_1\\0\end{pmatrix} & \text{on} \quad Q_3\\ \begin{pmatrix}-\psi_3\\0\end{pmatrix} & \text{on} \quad Q_4,\end{cases} \qquad \begin{pmatrix}0\\\Psi\end{pmatrix}=\begin{cases}\mathbf{0} & \text{on} \quad Q_1\\ \begin{pmatrix}0\\\varphi_3\end{pmatrix} & \text{on} \quad Q_2\\ \begin{pmatrix}0\\\varphi_1\end{pmatrix} & \text{on} \quad Q_3\\ \mathbf{0} & \text{on} \quad Q_4.\end{cases}$$

2.4 The inf-sup condition for the two coupled problems

In this section we will show that the pair of spaces $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$ and $(\mathbf{V}^h, M^h)$ satisfy the inf-sup condition. First we will show the inf-sup condition for the Stokes-Brinkman problem.

Theorem 2.4.1. *The pair of finite element spaces $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$ satisfies the inf-sup condition, that is, there exists a positive constant β independent of h such that*

$$\sup_{\mathbf{v}_h\in[\mathcal{NC}_0^h]^2}\frac{b_h(\mathbf{v}_h,q_h)}{|||\mathbf{v}_h|||_h}\geq\beta\|q_h\|_{0,\Omega}\quad\forall q_h\in\mathcal{P}_{cf}^h.$$

Proof. It is enough to show that $\nu_2=1$ case since $|||\mathbf{v}_h|||_h$ decreases as ν_2 decreases. Let $\widehat{\mathcal{T}}_h$ be the triangulation of Ω into triangles by dividing each quadrilateral into two triangles. Consider the P_1–nonconforming finite element space $\widehat{\mathcal{NC}}_0^h$ on $\widehat{\mathcal{T}}_h$. We then easily observe that $\mathcal{NC}_0^h\subset\widehat{\mathcal{NC}}_0^h$. Since the space $\widehat{\mathcal{NC}_0^h}$ satisfies discrete Poincaré–inequality [47], the subspace $\mathcal{NC}_0^h$ is also satisfied. Hence, we get

$$|||\mathbf{v}_h|||_h \leq M^{1/2}\|\mathbf{v}_h\|_{1,h} \leq C|\mathbf{v}_h|_{1,h},$$

where $M = \max(\nu_1, 1)$. For the pair of spaces $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$ equipped with the H^1–seminorm on velocity fields, the inf-sup condition for the standard Stokes problem is shown in [32]. Summarizing the above, this completes the proof. $\square$

Next, we are going to prove the inf-sup condition for the pair of spaces $(\mathbf{V}^h, M^h)$. To this end, let

$$\begin{aligned}
\widetilde{\mathbf{X}}_1^h &= \left\{\mathbf{v} \in \mathbf{X}_1^h \;\middle|\; \mathbf{v}(M_j) = 0 \text{ at the midpoint of each edge in } \Gamma_I \right\},\\
\widetilde{\mathbf{X}}_2^h &= \left\{\mathbf{v} \in \mathbf{X}_2^h \;\middle|\; \mathbf{v} \cdot \widehat{\mathbf{n}} = 0 \text{ on } \Gamma_I\right\},\\
\widetilde{M}_{1,cf}^h &= \left\{q_h \in M_{1,cf}^h \;\middle|\; \int_{\Omega_1} q_h \, d\mathbf{x} = 0\right\},\\
\widetilde{M}_2^h &= \left\{q_h \in M_2^h \;\middle|\; \int_{\Omega_2} q_h \, d\mathbf{x} = 0\right\},
\end{aligned}$$

and $\mathcal{T}_{h,1}^M$ denote the macro triangulation such that for each $Q_M \in \mathcal{T}_{h,1}^M$ is composed of four elements $Q_j,\ j = 1, \cdots, 4$. Define the operator $\Pi_M : H_{\Gamma_1}^1(\Omega_1) \to \mathcal{NC}_{\Gamma_1}^h$ satisfying

$$\int_{\Gamma_M} v \, d\sigma = \int_{\Gamma_M} \Pi_M v \, d\sigma \quad \text{ for all edges } \Gamma_M \text{ of } Q_M \in \mathcal{T}_{h,1}^M, \tag{2.14}$$

if $v \in H_{\Gamma_1}^1(\Omega_1) := \{v \in H^1(\Omega_1) : v = 0 \text{ on } \Gamma_1\}$. Indeed, for given $v \in H_{\Gamma_1}^1(\Omega_1)$, define the values of $\Pi_M v \in \mathcal{NC}_{\Gamma_1}^h$ as follows:

$$\Pi_M v(\text{midpoint of } E) = \oint_{\Gamma_M} v \, d\sigma,$$

if the edge $E \in \mathcal{T}_h$ meets the midpoint of macro edge $\Gamma_M \in \mathcal{T}_{h,1}^M$. By proposition 3.1 in [40], we have

$$|\Pi_M v|_{1,h,\Omega_1} \leq C|v|_{1,\Omega_1}. \tag{2.15}$$

Moreover, for any $q_h = (q_{h,1}, q_{h,2}) \in M^h$, it can be orthogonally split as

$$q_{h,j} = \widetilde{q}_{h,j} + \widehat{q}_{h,j}, \quad j = 1, 2,$$

where $\widetilde{q}_{h,1} \in \widetilde{M}_{1,cf}^h$, $\widetilde{q}_{h,2} \in \widetilde{M}_2^h$ and $\widehat{q}_{h,j} = \frac{1}{|\Omega_j|}(q_{h,j}, 1)_{\Omega_j}$. From the orthogonality, we have

$$\|q_{h,j}\|_{0,\Omega_j}^2 = \|\widetilde{q}_{h,j}\|_{0,\Omega_j}^2 + \|\widehat{q}_{h,j}\|_{0,\Omega_j}^2,$$

and

$$\widehat{q}_{h,1}|\Omega_1| + \widehat{q}_{h,2}|\Omega_2| = 0. \tag{2.16}$$

Also let $\widehat{M}^h$ be the subspace M^h defined by

$$\widehat{M}^h = \left\{ \widehat{q}_h = (\widehat{q}_{h,1}, \widehat{q}_{h,2}) \in M^h \;\middle|\; \widehat{q}_{h,j} \in \mathbb{R}, \quad j = 1, 2 \right\}.$$

Since the condition (2.16), the L^2–norm on $\widehat{M}^h$ is given by

$$\|\widehat{q}_h\|_0 = \left(\sum_{j=1}^{2} \widehat{q}_{h,j}^2 |\Omega_j| \right)^{1/2} = \widehat{q}_{h,2} \left(\frac{|\Omega_2|^2}{|\Omega_1|} + |\Omega_2| \right)^{1/2}.$$

In the following, we show the pair of spaces $(\mathbf{V}^h, \widehat{M}^h)$ satisfies the inf-sup condition. In the similar direction of [8], one can construct a continuous function $w \in H^1_{\Gamma_1}(\Omega_1)$, such that

$$\int_{\Gamma_I} w \, d\sigma = 1 \quad \text{and} \quad \|w\|_{1,\Omega_1} \leq \frac{C}{|\Gamma_I|}. \tag{2.17}$$

Denote $\mathbf{w}_{h,1} = \Pi_M \begin{pmatrix} w \\ w \end{pmatrix} \in \mathbf{X}_1^h$. By the properties (2.14), (2.15) and (2.17), we have

$$\int_{\Gamma_I} \mathbf{w}_{h,1} \cdot \widehat{\mathbf{n}} \, d\sigma = n_x + n_y \quad \text{and} \quad |\mathbf{w}_{h,1}|_{1,h,\Omega_1} \leq \frac{C}{|\Gamma_I|}. \tag{2.18}$$

In the porous medium domain Ω_2, we define a discrete Darcy harmonic extension function $\mathbf{w}_{h,2} \in \mathbf{X}_2^h$ such that for any $E \in \mathcal{E}_h(\Gamma_I)$, $\langle (\mathbf{w}_{h,1} - \mathbf{w}_{h,2}) \cdot \widehat{\mathbf{n}}, 1 \rangle_E = 0$, then it holds

$$\int_{\Gamma_I} \mathbf{w}_{h,2} \cdot \widehat{\mathbf{n}} \, d\sigma = n_x + n_y \quad \text{and} \quad \|\mathbf{w}_{h,2}\|_{\mathrm{div},\Omega_2} \leq \frac{C}{|\Gamma_I|}. \tag{2.19}$$

For any $\widehat{q}_h = (\widehat{q}_{h,1}, \widehat{q}_{h,2}) \in \widehat{M}^h$, define

$$\mathbf{v}_k = -|\Omega_k| \widehat{q}_{h,k} \left[\mathbf{w}_{h,1} \chi_{\Omega_1} + \mathbf{w}_{h,2} \chi_{\Omega_2} \right], \quad k = 1, 2. \tag{2.20}$$

Then, we have

$$\langle (\mathbf{v}_k|_{\Omega_1} - \mathbf{v}_k|_{\Omega_2}) \cdot \widehat{\mathbf{n}}, 1 \rangle_E = -|\Omega_k| \widehat{q}_{h,k} \langle (\mathbf{w}_{h,1} - \mathbf{w}_{h,2}) \cdot \widehat{\mathbf{n}}, 1 \rangle_E = 0$$

for any $E \in \mathcal{E}_h(\Gamma_I)$. Thus, $\mathbf{v}_j \in \mathbf{V}^h$, $j = 1, 2$. And from the definitions of (2.20) and the properties (2.18) and (2.19), we get for $j = 1, 2$

$$\begin{aligned} \int_{\Gamma_I} \mathbf{v}_j|_{\Omega_1} \cdot \widehat{\mathbf{n}} = \int_{\Gamma_I} \mathbf{v}_j|_{\Omega_2} \cdot \widehat{\mathbf{n}} &= -(n_x + n_y) \widehat{q}_{h,j} |\Omega_j|, \\ \|\mathbf{v}_j\|_{\mathbf{X}^h} &\leq C \|\widehat{q}_{h,j}\|_{0,\Omega_j}. \end{aligned}$$

Summarizing the above, we have the following lemma by using same argument in [30].

Lemma 2.4.2. *The pair of finite element spaces $(\mathbf{V}^h, \widehat{M}^h)$ satisfies the inf-sup condition, that is, there exists a positive constant β independent of h such that*

$$\sup_{\mathbf{v}_h \in \mathbf{V}^h} \frac{b_h(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{\mathbf{X}^h}} \geq \beta \|q_h\|_{0,\Omega} \quad \forall q_h \in \widehat{M}^h.$$

From [32, 43], we have that the spaces $(\widetilde{\mathbf{X}}_1^h, \widetilde{M}_{1,cf}^h)$ and $(\widetilde{\mathbf{X}}_2^h, \widetilde{M}_2^h)$ are locally div-stable in each subdomain. By the Lemma 2.4.2 and the technique called subspace theorem derived by Qin [41], we have the following theorem.

Theorem 2.4.3. *The pair of finite element spaces $(\mathbf{V}^h, M^h)$ satisfies the inf-sup condition, that is, there exists a positive constant β independent of h such that*

$$\sup_{\mathbf{v}_h \in \mathbf{V}^h} \frac{b_h(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{\mathbf{X}^h}} \geq \beta \|q_h\|_{0,\Omega} \quad \forall q_h \in M^h.$$

2.5 The Interpolation operator and convergence analysis

In this section, error estimates in the (broken) energy norm for the velocity and the L^2–norm for the pressure will be derived. The discrete inf-sup condition will be used to estimate the error of the pressure approximation. We first focus on the pair of spaces $([\mathcal{NC}_0^h]^2, \mathcal{P}_{cf}^h)$ to approximate the Stokes-Brinkman problem.

Define interpolation operator $R_h : [H^2(\Omega)]^2 \cap [H_0^1(\Omega)]^2 \rightarrow [\mathcal{NC}_0^h]^2$ and

$S_h : H^1(\Omega) \cap L_0^2(\Omega) \to \mathcal{P}_{cf}^h$ by

$$
\begin{aligned}
R_h \mathbf{w}(M_k^Q) &= \frac{\mathbf{w}(V_{k-1}^Q) + \mathbf{w}(V_k^Q)}{2} \quad && \forall Q \in \mathcal{T}_h, \\
(S_h q, z) &= (q, z) && \forall z \in \mathcal{P}_{cf}^h
\end{aligned}
$$

where V_{k-1}^Q and V_k^Q are the two vertices of the edge E_k^Q with the midpoint M_k^Q for each $Q \in \mathcal{T}_h$. Then the standard polynomial approximation results imply that

$$
\begin{aligned}
&\|\mathbf{v} - R_h\mathbf{v}\| + h\left(\sum_{Q_j \in \mathcal{T}_h} |\mathbf{v} - R_h\mathbf{v}|_{1,j}^2\right)^{1/2} + h^2\left(\sum_{Q_j \in \mathcal{T}_h} |\mathbf{v} - R_h\mathbf{v}|_{2,j}^2\right)^{1/2} \\
&\quad + h^{1/2}\left(\sum_{Q_j \in \mathcal{T}_h} ||\mathbf{v} - R_h\mathbf{v}||_{L^2(\partial Q_j)}^2\right)^{1/2} \le Ch^2\|\mathbf{v}\|_2, \quad \forall \mathbf{v} \in [H^2(\Omega)]^2 \cap [H_0^1(\Omega)]^2, \\
&\|q - S_h q\|_{0,\Omega} \le Ch\|q\|_1, \quad \forall q \in H^1(\Omega) \cap L_0^2(\Omega).
\end{aligned}
$$

The error estimates for the Stokes-Brinkman problem can be derived by a slight modification in [9, 21]. Also the use of a duality argument is analogous to that in [9], and therefore we omit the details. Finally we have the following theorem.

Theorem 2.5.1. *Assume that* (2.1) *and* (2.2) *is* H^2*–regular. Let* $(\mathbf{u}, p)$ *and* $(\mathbf{u}_h, p_h)$ *be the solutions of* (2.8) *and* (2.12), *respectively. Then the following optimal-order error estimate holds:*

$$
\|\mathbf{u} - \mathbf{u}_h\|_0 + h\,[|||\mathbf{u} - \mathbf{u}_h|||_h + \|p - p_h\|_0] \le Ch^2(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}).
$$

On the other hand, the error estimates for the Stokes-Darcy problem are not obvious. Since we consider the pair of spaces $(\mathbf{V}^h, M^h)$ to approximate the weak formulation (2.11), we need the interpolation operator I^h corresponding to $\mathbf{V}^h$, that is, the approximation properties of

$$\mathbf{V}^h = \Big\{(\mathbf{v}_{h,1}, \mathbf{v}_{h,2}) \in \mathbf{X}_1^h \times \mathbf{X}_2^h \;\Big|\; \langle \mathbf{v}_{h,1} \cdot \widehat{\mathbf{n}}_1 + \mathbf{v}_{h,2} \cdot \widehat{\mathbf{n}}_2, \mu_h \rangle_{\Gamma_I} = 0 \quad \text{for all } \mu_h \in \Lambda^h \Big\}$$

must be described by a direct construction. Thus, we should construct an interpolation operator

$$I_v^h := \mathbf{W} \to \mathbf{V}^h,$$

where $\mathbf{W}$ denotes the subspace of $\mathbf{V}$ of sufficiently smooth functions defined by

$$\begin{aligned} \mathbf{W} := \{\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{X} \ \mid \ \mathbf{v}_j \in \mathbf{W}_j := \mathbf{X}_j \cap [H^2(\Omega_j)]^2,\ j = 1,2 \\ \text{and } \ \mathbf{v}_1 \cdot \widehat{\mathbf{n}}_1|_{\Gamma_I} = \mathbf{v}_2 \cdot \widehat{\mathbf{n}}_1|_{\Gamma_I} \text{ in } L^2(\Gamma_I)\}. \end{aligned}$$

Let $I_{NC}^h : \mathbf{W}_1 \to \mathbf{X}_1^h$ and $I_{RT}^h : \mathbf{W}_2 \to \mathbf{X}_2^h$ denote the P_1–nonconforming and usual Raviart-Thomas interpolation operator, respectively. Then $I_v^h = (I_1^h \mathbf{v}, I_2^h \mathbf{v}) \in \mathbf{V}^h$ defined by

$$I_1^h \mathbf{v} := I_{NC}^h \mathbf{v}_1 \in \mathbf{X}_1^h \qquad \text{and} \qquad I_2^h \mathbf{v} := I_{RT}^h \mathbf{v}_2 - \delta^h \in X_2^h,$$

where the correction term δ^h will be constructed to enforce $I_v^h \mathbf{v}$ belong to $\mathbf{V}^h$. By the definition of I_{RT}^h and Λ^h, we get the following relation for all $\mu_h \in \Lambda^h$

$$\begin{aligned} \Big\langle I_1^h \mathbf{v} \cdot \widehat{\mathbf{n}}_1 + I_2^h \mathbf{v} \cdot \widehat{\mathbf{n}}_2, \mu_h \Big\rangle_{\Gamma_I} &= -\Big\langle I_{NC}^h \mathbf{v}_1 \cdot \widehat{\mathbf{n}}_2, \mu_h \Big\rangle_{\Gamma_I} + \langle \mathbf{v}_2 \cdot \widehat{\mathbf{n}}_2, \mu_h \rangle_{\Gamma_I} - \Big\langle \delta^h \cdot \widehat{\mathbf{n}}_2, \mu_h \Big\rangle_{\Gamma_I} \\ &= \Big\langle (\mathbf{v}_1 - I_{NC}^h \mathbf{v}_1) \cdot \widehat{\mathbf{n}}_2, \mu_h \Big\rangle_{\Gamma_I} - \Big\langle \delta^h \cdot \widehat{\mathbf{n}}_2, \mu_h \Big\rangle_{\Gamma_I}. \end{aligned} \tag{2.21}$$

In order to construct δ^h, we will define $\delta \in \mathbf{X}_2 \cap [H^1(\Omega_2)]^2$ such that

$$\delta = \mathbf{v}_1 - I_C^h \mathbf{v}_1 \text{ on } \Gamma_I, \quad \text{and} \quad \|\delta\|_{1,\Omega_2} \leq C|\mathbf{v}_1 - I_C^h \mathbf{v}_1|_{1,\Omega_1},$$

where I_C^h denotes the usual bilinear interpolation operator. To this end, let $\mathbf{z}$ be the extension function $\mathbf{v}_1 - I_C^h \mathbf{v}_1$ by zero on Ω_2, *i.e.*,

$$\mathbf{z} := \begin{cases} \mathbf{v}_1 - I_C^h \mathbf{v}_1 & \text{on} \quad \Gamma_I, \\ \mathbf{0} & \text{on} \quad \partial\Omega_2 \backslash \Gamma_I. \end{cases}$$

By the definition of interpolation operator I_C^h, $\mathbf{v}_1 - I_C^h \mathbf{v}_1 = 0$ on $\partial\Gamma_I$, $\mathbf{v}_1 - I_C^h \mathbf{v}_1 \in [H_{00}^{1/2}(\Gamma_I)]^2$, so $\mathbf{z} \in [H^{1/2}(\Omega_2)]^2$. Applying the usual trace theorem, discrete Poincaré-Friedrichs inequality and (2.10), we have

$$\begin{aligned} \|\mathbf{z}\|_{1/2,\partial\Omega_2} &\leq \|\mathbf{v}_1 - I_C^h \mathbf{v}_1\|_{1/2,\Gamma_I} \leq \|\mathbf{v}_1 - I_C^h \mathbf{v}_1\|_{1/2,\partial\Omega_1} \\ &\leq \|\mathbf{v}_1 - I_C^h \mathbf{v}_1\|_{1,\Omega_1} \leq |\mathbf{v}_1 - I_C^h \mathbf{v}_1|_{1,\Omega_1}. \end{aligned}$$

Since $[H^{1/2}(\partial\Omega_2)]^2$ be the trace on $[H^1(\Omega_2)]^2$, we can choose $\delta \in [H^1(\Omega_2)]^2$ extending $\mathbf{z}$ onto Ω_2 such that

$$\|\delta\|_{1,\Omega_2} \leq C\|\mathbf{z}\|_{1/2,\partial\Omega_2} \leq C|\mathbf{v}_1 - I_C^h \mathbf{v}_1|_{1,\Omega_1}.$$

Now we define δ^h as follows:

$$\delta^h := I_{RT}^h \delta$$

The property of the interpolation I_{RT}^h and definition of δ give that for $\mu_h \in \Lambda^h$

$$\left\langle \delta^h \cdot \widehat{\mathbf{n}}_2, \mu_h \right\rangle_{\Gamma_I} = \langle \delta \cdot \widehat{\mathbf{n}}_2, \mu_h \rangle_{\Gamma_I} = \left\langle (\mathbf{v}_1 - I_C^h \mathbf{v}_1) \cdot \widehat{\mathbf{n}}_2, \mu_h \right\rangle_{\Gamma_I}.$$

We note that $I^h_{NC}\mathbf{v}$ and $I^h_C\mathbf{v}$ are linear function on each edge and have same value on midpoint. Combining this with (2.21) gives for all $\mu_h \in \Lambda^h$

$$\begin{aligned}\left\langle I^h_1\mathbf{v}\cdot\widehat{\mathbf{n}}_1 + I^h_2\mathbf{v}\cdot\widehat{\mathbf{n}}_2, \mu_h\right\rangle_{\Gamma_I} &= \left\langle (I^h_C\mathbf{v}_1 - I^h_{NC}\mathbf{v}_1)\cdot\widehat{\mathbf{n}}_2, \mu_h\right\rangle_{\Gamma_I}\\ &= 0.\end{aligned}$$

This implies that $I^h_v = (I^h_1\mathbf{v}, I^h_2\mathbf{v}) \in \mathbf{V}^h$. Next we will observe interpolation error estimate for I^h. To this end, we shall need an estimate the correction term δ^h.

From the interpolation error estimates [21] for I^h_{RT}, we get for each $Q \in \mathcal{T}_{h,2}$

$$\|\delta^h\|_{1,Q} \le \|\delta\|_{1,Q} + \|\delta - I^h_{RT}\delta\|_{1,Q} \le \|\delta\|_{1,Q}.$$

Moreover,

$$\|\delta^h\|_{\mathbf{X}_2} = \left\{\|\delta^h\|^2_{0,\Omega_2} + \|\nabla\cdot\delta^h\|^2_{0,\Omega_2}\right\}^{1/2} \le \left\{\sum_{Q\in\mathcal{T}_{h,2}} \|\delta^h\|^2_{1,Q}\right\}^{1/2},$$

which implies

$$\|\delta^h\|_{\mathbf{X}_2} \le \|\delta\|_{1,\Omega_2} \le C|\mathbf{v}_1 - I^h_C\mathbf{v}_1|_{1,\Omega_1}. \tag{2.22}$$

By the error estimate for each interpolation operator and (2.22), the boundedness for the interpolation operator I^h are given by

$$\begin{aligned}\|\mathbf{v} - I^h_v\mathbf{v}\|_{\mathbf{X}^h} &\le |\mathbf{v}_1 - I^h_1\mathbf{v}_1|_{1,h,\Omega_1} + \|\mathbf{v}_2 - I^h_2\mathbf{v}_2\|_{\mathbf{X}_2}\\ &\le |\mathbf{v}_1 - I^h_{NC}\mathbf{v}_1|_{1,h,\Omega_1} + \|\mathbf{v}_2 - I^h_{RT}\mathbf{v}_2\|_{\mathbf{X}_2} + \|\delta^h\|_{\mathbf{X}_2}\\ &\le |\mathbf{v}_1 - I^h_{NC}\mathbf{v}_1|_{1,h,\Omega_1} + \|\mathbf{v}_2 - I^h_{RT}\mathbf{v}_2\|_{\mathbf{X}_2} + C|\mathbf{v}_1 - I^h_C\mathbf{v}_1|_{1,\Omega_1}\\ &\le Ch\left\{\|\mathbf{v}_1\|_{2,\Omega_1} + \|\mathbf{v}_2\|_{1,\Omega_2} + \|\nabla\cdot\mathbf{v}_2\|_{1,\Omega_2}\right\}.\end{aligned} \tag{2.23}$$

For the pressure fields, we define interpolation operator $I_p^h : H^1(\Omega) \cap L_0^2(\Omega) \rightarrow M^h$ by

$$(I_p^h q, z) = (q, z) \quad \forall z \in M^h. \tag{2.24}$$

Then we obtain

$$\|q - I_p^h q\|_{0,\Omega} \leq Ch\|q\|_1, \quad \forall q \in H^1(\Omega) \cap L_0^2(\Omega).$$

For deriving the error estimate of the proposed finite element method, we shall bound the approximation error and consistency error separately. Let denote $\mathcal{E}_{h,1}$ be the set of edges of elements on $\mathcal{T}_{h,1}$ which is not on $\partial\Omega_1$ and $[\cdot]_e$ be the jump through E. The abstract error estimate for mixed finite element formulation gives the following lemma [7, 21].

Lemma 2.5.2. *Let* $(\mathbf{u}, p)$ *and* $(\mathbf{u}_h, p_h)$ *be the solutions of* (2.11) *and* (2.13), *respectively. Then the following error estimate holds:*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{X}^h} + \|p - p_h\|_0 \quad \leq \quad & \Bigg\{ \inf_{\mathbf{v}_h \in \mathbf{V}^h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{X}^h} + \inf_{q_h \in M^h} \|p - q_h\|_0 \\ & + \sup_{\mathbf{v}_h \in \mathbf{V}^h} \frac{\left| \sum_{E \in \mathcal{E}_{h,1}} \left\langle \frac{\partial \mathbf{w}}{\partial \widehat{\mathbf{n}}}, [\mathbf{v}_h] \right\rangle_E - \sum_{E \in \mathcal{E}_{h,1}} \langle q_1, [\widehat{\mathbf{n}} \cdot \mathbf{v}_h] \rangle \right|}{\|\mathbf{v}_h\|_{\mathbf{X}^h}} \\ & + \sup_{\mathbf{v}_h \in \mathbf{V}^h} \frac{\left| \sum_{E \in \mathcal{E}_h(\Gamma_I)} \langle q_2, (\mathbf{v}_h|_{\Omega_1} - \mathbf{v}_h|_{\Omega_2}) \cdot \widehat{\mathbf{n}} \rangle_E \right|}{\|\mathbf{v}_h\|_{\mathbf{X}^h}} \Bigg\} \end{aligned} \tag{2.25}$$

To bound the two consistency error terms in (2.25), we will apply the following lemma.

Lemma 2.5.3. *For any* $\mathbf{v}_h \in \mathbf{V}^h$*, we have the following estimates:*

$$\left| \sum_{E\in\mathcal{E}_{h,1}} \left\langle \frac{\partial \mathbf{w}}{\partial \widehat{\mathbf{n}}}, [\mathbf{v}_h] \right\rangle_E \right| \leq Ch\|\mathbf{w}\|_{2,\Omega_1}\|\mathbf{v}_h\|_{\mathbf{X}^h} \qquad \forall \mathbf{w} \in [H^2(\Omega_1)]^2, \tag{2.26a}$$

$$\left| \sum_{E\in\mathcal{E}_{h,1}} \langle q_1, [\widehat{\mathbf{n}} \cdot \mathbf{v}_h] \rangle_E \right| \leq Ch\|q_1\|_{1,\Omega_1}\|\mathbf{v}_h\|_{\mathbf{X}^h} \qquad \forall q_1 \in H^1(\Omega_1), \tag{2.26b}$$

$$\left| \sum_{E\in\mathcal{E}_h(\Gamma_I)} \langle q_2, (\mathbf{v}_h|_{\Omega_1} - \mathbf{v}_h|_{\Omega_2}) \cdot \widehat{\mathbf{n}} \rangle_E \right| \leq Ch\|q_2\|_{1,\Omega_2}\|\mathbf{v}_h\|_{\mathbf{X}^h} \qquad \forall q_2 \in H^1(\Omega_2). \tag{2.26c}$$

Proof. The estimate of the consistency error (2.26a) and (2.26b) can be shown in a similar direction [9, 10]. It remains to prove (2.26c). Since $\mathbf{v}_h$ belongs to $\mathbf{V}^h$ and the normal trace of $\mathbf{v}_2$ on Γ_I belongs to Λ^h, we get

$$\begin{aligned} \sum_{E\in\mathcal{E}_h(\Gamma_I)} \langle q_2, (\mathbf{v}_h|_{\Omega_1} - \mathbf{v}_h|_{\Omega_2}) \cdot \widehat{\mathbf{n}} \rangle_E &= \sum_{E\in\mathcal{E}_h(\Gamma_I)} \langle q_2 - \widetilde{q}_2, (\mathbf{v}_h|_{\Omega_1} - \mathbf{v}_h|_{\Omega_2}) \cdot \widehat{\mathbf{n}} \rangle_E \\ &= \sum_{E\in\mathcal{E}_h(\Gamma_I)} \langle q_2 - \widetilde{q}_2, \mathbf{v}_h|_{\Omega_1} \cdot \widehat{\mathbf{n}} \rangle_E , \end{aligned}$$

where $\widetilde{q}_2 \in \Lambda^h$ is the local L^2-projection of the function q_2 on E. By the assumption on $\mathcal{T}_h$, for each edge E of $\mathcal{E}_h(\Gamma_I)$ is shared by the two elements $Q_j \in \mathcal{T}_{h,j}$ $j = 1, 2$. Define on element Q_1 the constant $\mathbf{m}_1$ as the local L^2-projection of the function $\mathbf{v}_h$. Then, from the constant mean value approximation and the trace inequality, we obtain

$$\begin{aligned} \sum_{E\in\mathcal{E}_h(\Gamma_I)} \langle q_2 - \widetilde{q}_2, \mathbf{v}_h|_{\Omega_1} \cdot \widehat{\mathbf{n}} \rangle_E &= \sum_{E\in\mathcal{E}_h(\Gamma_I)} \langle q_2 - \widetilde{q}_2, (\mathbf{v}_h|_{\Omega_1} - \mathbf{m}_1) \cdot \widehat{\mathbf{n}} \rangle_E \\ &\leq \sum_{E\in\mathcal{E}_h(\Gamma_I)} \|q_2 - \widetilde{q}_2\|_{0,E} \|\mathbf{v}_h|_{\Omega_1} - \mathbf{m}_1\|_{0,E} \\ &\leq Ch\|q_2\|_{1,\Omega_2}\|\mathbf{v}_h\|_{\mathbf{X}^h} \end{aligned}$$

This completes the proof. $\square$

Combining the Lemma 2.5.2 and Lemma 2.5.3, we conclude that the following optimal *a priori* error estimate.

Theorem 2.5.4. *Let* $(\mathbf{u}, p)$ *and* $(\mathbf{u}_h, p_h)$ *be the solutions of* (2.11) *and* (2.13), *respectively. Assuming that* $(\mathbf{u}_1, p_1) \in [H^2(\Omega_1)]^2 \times H^1(\Omega_1)$, $(\mathbf{u}_2, p_2) \in [H^1(\Omega_2)]^2 \times H^1(\Omega_2)$ *and* $\nabla \cdot \mathbf{u}_2 \in H^1(\Omega_2)$, *then we have*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{X}^h} + \|p - p_h\|_{0,\Omega} \leq Ch\left[(\|\mathbf{u}\|_{2,\Omega_1} + \|p\|_{1,\Omega_1}) + (\|\mathbf{u}\|_{1,\Omega_2} + \|\nabla \cdot \mathbf{u}\|_{1,\Omega_2} + \|p\|_{1,\Omega_2})\right].$$

2.6 Numerical results

Now we illustrate a numerical example for the Stokes-Brinkman and Stokes-Darcy problems for the domain $\Omega = [0,1]^2$. We present two examples illustrating the performance of our proposed nonconforming finite element method on a set of uniform triangulations of the domain. Throughout the numerical study, we fix $\nu_1 = \alpha = 1$. For the case of Stokes-Brinkman problem ($\nu_2 \in (0,1]$), we set $\Omega_1 = (0,1/2) \times (0,1)$, $\Omega_2 = (1/2,1) \times (0,1)$, and Γ_I is the line $x = 1/2$. It is difficult to construct solutions that satisfy the entire system. The main difficulty is finding a solution satisfying the interface conditions (2.3)-(2.4). We simply use the same technique of generalizing the equations to include a nonhomogeneous term. That is, we replace (2.4) by

$$(\nu_1 \nabla \mathbf{u}_1 - p_1 \mathbf{I}) \cdot \widehat{\mathbf{n}}_1 = (\nu_2 \nabla \mathbf{u}_2 - p_2 \mathbf{I}) \cdot \widehat{\mathbf{n}}_1 + \mathbf{h}_1 \qquad \text{on } \Gamma_I.$$

We first choose $\mathbf{u}$ satisfying (2.3), and then include the term $\langle \mathbf{h}_1, \mathbf{v} \rangle_{\Gamma_I}$ on the right side. The interface condition for the Stokes-Darcy problem can be also applied in a similar way. Choose exact solution $\mathbf{u}$ and p as follows.

$$\begin{aligned}
\mathbf{u}_1(x,y) &= \begin{pmatrix} s(x)s'(y) \\ s'(x)(y^2-y) \end{pmatrix}, \quad \mathbf{u}_2(x,y) = \begin{pmatrix} 100(x-\frac{1}{2})(x-1)^2 s(y) \\ (x-1)^2\left[2\pi(y^2-y)+(x-\frac{1}{2})\sin(2\pi y)\right] \end{pmatrix}, \\
p_1(x,y) &= \sin(\pi x) - \frac{2}{\pi}, \qquad p_2(x,y) = \sin(4\pi x)(y^2+1),
\end{aligned}$$

where $s(t) = \sin(2\pi t)(t^2-t)$, $s'(t)$ denotes its derivative. The numerical results are shown in Table 2.1-Table 2.2.

h	$\|\mathbf{u}-\mathbf{u}_h\|_0$	Order	$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}$	Order	$\|p-p_h\|_0$	Order
1/8	2.3134E-2	-	7.6436E-1	-	3.4863E-1	-
1/16	5.7629E-1	2.01	3.8827E-1	0.98	1.6154E-1	1.11
1/32	1.4407E-3	2.00	1.9498E-1	0.99	7.9365E-2	1.03
1/64	3.6020E-4	2.00	9.7596E-2	1.00	3.9509E-2	1.01
1/128	9.0046E-5	2.00	4.8812E-2	1.00	1.9732E-2	1.00

Table 2.1: Numerical result for the Stokes-Brinkman problem with $\nu_2 = 1$

h	$\|\mathbf{u}-\mathbf{u}_h\|_0$	Order	$\|\mathbf{u}-\mathbf{u}_h\|_{1,h}$	Order	$\|p-p_h\|_0$	Order
1/8	9.7591E-2	-	5.7744E-1	-	3.2229E-1	-
1/16	2.4401E-2	2.00	2.8900E-1	1.00	1.5859E-1	1.02
1/32	6.1523E-3	1.99	1.4456E-1	1.00	7.9004E-2	1.01
1/64	1.5481E-3	1.99	7.2287E-2	1.00	3.9463E-2	1.00
1/128	3.8840E-4	1.99	3.6144E-2	1.00	1.9727E-2	1.00

Table 2.2: Numerical result for the Stokes-Brinkman problem with $\nu_2 = 10^{-6}$

For the case of Stokes-Darcy problem ($\nu_2 = 0$), we will consider the case of porous medium is entirely enclosed within the fluid region, then $\Gamma_I = \partial\Omega_2$. Set $\Omega_2 = (1/4,3/4)\times(1/4,3/4)$ and $\Omega_1 = \Omega\backslash\bar{\Omega}_2$. We will bring the numerical example examined in [20] transformed to $[0,1]^2$, *i.e.*, exact solution $\mathbf{u}$ and p

are given by

$$
\begin{aligned}
\mathbf{u}_1(x,y) &= \begin{pmatrix} \pi\sin^2(\pi s(x))\cos^4(\pi s(x))\sin(\pi s(y))\cos^3(\pi s(y))(1-3\sin^2(\pi s(y)) \\ -\pi\sin^2(\pi s(y))\cos^4(\pi s(y))\sin(\pi s(x))\cos^3(\pi s(x))(1-3\sin^2(\pi s(x)) \end{pmatrix}, \\
\mathbf{u}_2(x,y) &= \frac{1}{4}\begin{pmatrix} -(2s(x)-1)(2s(x)+1)(20s(x)^2-4s(x)-1)) \\ -(2s(y)-1)(2s(y)+1)(20s(y)^2+4s(y)-1)) \end{pmatrix}, \\
p(x,y) &= -\left(s(x)^2-\frac{1}{4}\right)^2(-4s(x)+1)-\left(s(y)^2-\frac{1}{4}\right)^2(-4s(y)-1)\,,
\end{aligned}
$$

where $s(t) = 2t-1$. We remark that the solutions show very oscillating behaviours. The numerical results are shown in Table 2.3. Another domain type as examined in Stokes-Brinkman problem was performed with same exact solution. The errors, omitted here, have optimal order convergence also. We can observe from these tables that the convergence order are optimal which confirms our theoretical analysis.

h	$\lvert\mathbf{u}_1-\mathbf{u}_{1,h}\rvert_{1,h,\Omega_1}$	Order	$\lVert\mathbf{u}_2-\mathbf{u}_{2,h}\rVert_{\mathrm{div},\Omega_2}$	Order	$\lVert p-p_h\rVert_0$	Order
1/8	1.2245E+0	-	1.8997E+0	-	5.4079E-1	-
1/16	7.8637E-1	0.64	1.0331E+0	0.88	3.0912E-1	0.81
1/32	4.1734E-1	0.91	5.2689E-1	0.97	1.2841E-1	1.27
1/64	2.1149E-1	0.98	2.6473E-1	0.99	5.8471E-2	1.14
1/128	1.0609E-1	1.00	1.3253E-1	1.00	2.8416E-2	1.04

Table 2.3: Numerical result for the Stokes-Darcy problem

Chapter 3

The three-dimensional cheapest finite element method for the Stokes problem

3.1 Introduction

This chapter is an extension of the recent works on a cheapest nonconforming finite element for the Stokes problem [38]. In order to solve the Stokes equations (and the Navier-Stokes equations) we attempt to apply the vector space of P_1-nonconforming finite element and the space of piecewise constant function space in approximating the velocity and pressure fields, respectively. Unfortunately, it appears that this pair is not stable for solving the problem. Instead of adding bubble functions for all elements, which is one of traditional stabilization schemes, we will show that adding a suitable number of global nonconforming bubble functions [15, 42] results in a stable Stokes problem if the mesh size is small enough.

The aim of this paper is to extend the idea of [38] to three dimensions. In contrast with [11, 38], which need only one global bubble-vector function to stable Stokes problem on two dimensions, three dimensional problem needs a suitable number of global bubble-vector functions. We will verify how many global bubble-vector functions do it take to construct a stable problem, about where to take it.

The outline of this paper is organized as follows. In Section 2, the P_1-nonconforming hexahedral finite element method for the three dimensional Stokes problem is introduced. In Section 3, we will show that the proposed element pair is singular and then the enrichment of the velocity space by adding the bubble function space will be introduced. Also, we will explain how to take bubble functions in given meshes. And then, the inf-sup condition for the discrete analogue of the bilinear form $b(\cdot,\cdot)$ is demonstrated. Finally, some numerical experiments for the three dimensional Stokes problem on various bubble function's positions are shown in Section 4.

3.2 The Stokes problem and P_1-nonconforming hexahedral finite element method

In this section we will discuss the simplest three dimensional nonconforming approximation of the incompressible Stokes problem. The velocity field will be approximated by the P_1-nonconforming hexahedral finite element space while the pressure will be approximated by piecewise constant.

3.2.1 The finite element formulation of the Stokes problem

In this paper, we assume that Ω is a hexahedron with boundary $\partial\Omega$. We consider the following stationary incompressible Stokes equations:

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

where $\mathbf{u} = (u_1, u_2, u_3)^T$ represents the velocity vector, p the pressure, $\mathbf{f} = (f_1, f_2, f_3)^T \in [H^{-1}(\Omega)]^3$ the body force, and $\nu > 0$ the viscosity. Set

$$L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q \, d\mathbf{V} = 0\}.$$

Here, and in what follows, we use the standard notations and definitions for the Sobolev spaces $H^s(K)$, and their associated inner products $(\cdot,\cdot)_{s,K}$, norms $||\cdot||_{s,K}$, and semi-norms $|\cdot|_{s,K}$. We will omit the subscripts s, K if $s = 0$ and $K = \Omega$. Then, the weak formulation of (3.1) is to find a pair $(\mathbf{u}, p) \in [H_0^1(\Omega)]^3 \times L_0^2(\Omega)$ such that

$$a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = (\mathbf{f},\mathbf{v}) \quad \forall\, \mathbf{v} \in [H_0^1(\Omega)]^3, \tag{3.2a}$$

$$b(\mathbf{u},q) = 0 \quad \forall\, q \in L_0^2(\Omega), \tag{3.2b}$$

where the bilinear forms $a(\cdot,\cdot) : [H_0^1(\Omega)]^3 \times [H_0^1(\Omega)]^3 \to \mathbb{R}$ and $b(\cdot,\cdot) : [H_0^1(\Omega)]^3 \times L_0^2(\Omega) \to \mathbb{R}$ are defined by

$$a(\mathbf{u},\mathbf{v}) = \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) \quad \text{and} \quad b(\mathbf{v},q) = -(\nabla\cdot\mathbf{v}, q).$$

Let $(\mathcal{T}_h)_{0<h<1}$ be a family of quasi-regular partitions of $\Omega \subset \mathbb{R}^3$ into disjoint

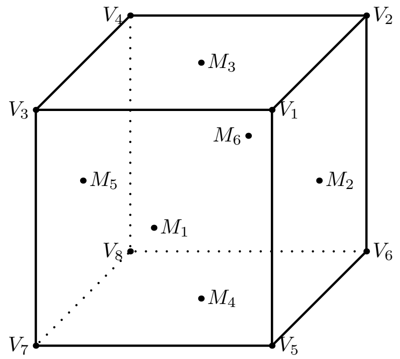


Figure 3.1: On a hexahedron, V_j denotes the vertices, $j = 1, 2, \cdots, 8$, and M_k denotes the centroid of faces F_k, $k = 1, \cdots, 6$.

hexahedrons $Q_j, j = 1, \cdots, N_Q$, where N_Q is the number of hexahedrons in $\mathcal{T}_h$. By using the P_1-nonconforming finite element on a hexahedral Q, we present the structure of the space and endow it with suitable degrees of freedom.

3.2.2 The P_1-nonconforming hexahedral element method

Let Q denote a hexahedron with the faces F_k and its the centroid M_k, $1 \leq k \leq 6$, (See Figure 3.1). Set

$$P_1(Q) = \text{Span}\{1, x, y, z\}.$$

We recall the following elementary, simple, but useful lemma:

Lemma 3.2.1 ([39]). *If $u \in P_1(Q)$, then $u(M_1) + u(M_6) = u(M_2) + u(M_5) = u(M_3) + u(M_4)$. Conversely, if u_j, for $j = 1, \cdots, 6$, are given values at M_j and satisfied $u_1 + u_6 = u_2 + u_5 = u_3 + u_4$, then there exists a unique function $u \in P_1(Q)$ such that $u(M_j) = u_j$, $j = 1, \cdots, 6$.*

For $j = 1, \cdots, 8$, let $\varphi_j : Q \to \mathbb{R}$ be the linear function defined by

$$
\begin{aligned}
\varphi_1(m) &= \begin{cases} 1, & \text{if } m = M_1, M_2, M_3, \\ 0, & \text{if } m = M_4, M_5, M_6, \end{cases} & \varphi_2(m) &= \begin{cases} 1, & \text{if } m = M_2, M_3, M_6, \\ 0, & \text{if } m = M_1, M_4, M_5, \end{cases} \\
\varphi_3(m) &= \begin{cases} 1, & \text{if } m = M_1, M_3, M_5, \\ 0, & \text{if } m = M_2, M_4, M_6, \end{cases} & \varphi_4(m) &= \begin{cases} 1, & \text{if } m = M_3, M_5, M_6, \\ 0, & \text{if } m = M_1, M_2, M_4, \end{cases} \\
\varphi_5(m) &= \begin{cases} 1, & \text{if } m = M_1, M_2, M_4, \\ 0, & \text{if } m = M_3, M_5, M_6, \end{cases} & \varphi_6(m) &= \begin{cases} 1, & \text{if } m = M_2, M_4, M_6, \\ 0, & \text{if } m = M_1, M_3, M_5, \end{cases} \\
\varphi_7(m) &= \begin{cases} 1, & \text{if } m = M_1, M_4, M_5, \\ 0, & \text{if } m = M_2, M_3, M_6, \end{cases} & \varphi_8(m) &= \begin{cases} 1, & \text{if } m = M_4, M_5, M_6, \\ 0, & \text{if } m = M_1, M_2, M_3. \end{cases}
\end{aligned} \tag{3.3}
$$

For each face-centroid M_j, one sees that $\varphi_1 + \varphi_6 = \varphi_2 + \varphi_5 = \varphi_3 + \varphi_4$. According to Lemma 3.2.1, we get

$$\mathcal{S}(Q) = \mathrm{Span}\{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6, \varphi_7, \varphi_8\},$$

whose dimension is four. Also, from (3.3), it follows that

$$
\begin{aligned}
\int_{F_j} \varphi_1 \, d\mathbf{x} &= \begin{cases} |F_j|, & j = 1, 2, 3, \\ 0, & j = 4, 5, 6, \end{cases} & \int_{F_j} \varphi_2 \, d\mathbf{x} &= \begin{cases} |F_j|, & j = 2, 3, 6, \\ 0, & j = 1, 4, 5, \end{cases} \\
\int_{F_j} \varphi_3 \, d\mathbf{x} &= \begin{cases} |F_j|, & j = 1, 3, 5, \\ 0, & j = 2, 4, 6, \end{cases} & \int_{F_j} \varphi_4 \, d\mathbf{x} &= \begin{cases} |F_j|, & j = 3, 5, 6, \\ 0, & j = 1, 2, 4, \end{cases} \\
\int_{F_j} \varphi_5 \, d\mathbf{x} &= \begin{cases} |F_j|, & j = 1, 2, 4, \\ 0, & j = 3, 5, 6, \end{cases} & \int_{F_j} \varphi_6 \, d\mathbf{x} &= \begin{cases} |F_j|, & j = 2, 4, 6, \\ 0, & j = 1, 3, 5, \end{cases} \\
\int_{F_j} \varphi_7 \, d\mathbf{x} &= \begin{cases} |F_j|, & j = 1, 4, 5, \\ 0, & j = 2, 3, 6, \end{cases} & \int_{F_j} \varphi_8 \, d\mathbf{x} &= \begin{cases} |F_j|, & j = 4, 5, 6, \\ 0, & j = 1, 2, 3, \end{cases}
\end{aligned}
$$

where $|F_j|$ denotes the area of the face F_j on hexahedron Q. Any four of these eight φ_j, $j = 1, \cdots, 8$, form the local basis functions for the P_1-nonconforming hexahedral element space.

The global P_1-nonconforming hexahedral finite element space is defined by

$$\begin{aligned}\mathcal{N}C_0^h(\Omega) = \{v \in L^2(\Omega) :\ & v_j := v|_{Q_j} \in P_1(Q_j)\ \forall j = 1, \cdots, N_Q; \\ & v \text{ is continuous at each interior face-centroid} \\ & \quad \text{and vanishes at each boundary face-centroid in } \mathcal{T}_h\}.\end{aligned}$$

The global basis functions of $\mathcal{NC}_0^h(\Omega)$ can be defined for each vertex as follows: For each interior vertex V_j of $\mathcal{T}_h$, let $\varphi_j \in \mathcal{NC}_0^h(\Omega)$ be defined such that it has value 1 at each interior face-centroid whose corner points contain the vertex V_j and value 0 at every other face-centroid in $\mathcal{T}_h$. Then, every $\mathbf{v}_h \in [\mathcal{NC}_0^h]^3$ is represented by

$$\mathbf{v}_h = \sum_{j=1}^{N_V^i} \begin{pmatrix} a_j \\ b_j \\ c_j \end{pmatrix} \varphi_j, \qquad \begin{pmatrix} a_j \\ b_j \\ c_j \end{pmatrix} \in \mathbb{R}^3,$$

where N_V^i denotes the number of interior vertices in $\mathcal{T}_h$. In addition, we consider the finite element space for the pressure variable

$$P^h = \{q \in L_0^2(\Omega) : q|_{Q_j} \in P_0(Q_j)\ \forall j = 1, \cdots, N_Q\},$$

where $P_0(A)$ denotes the space of piecewise constants on the set A.

For simplicity, let $(\cdot, \cdot)_j = (\cdot, \cdot)_{0,Q_j}$. We are now in a position to define the discrete weak form of (3.2) to seek a pair $(\mathbf{u}_h, p_h) \in [\mathcal{NC}_0^h]^3 \times P^h$ such that

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in [\mathcal{NC}_0^h]^3, && (3.4a) \\ b_h(\mathbf{u}_h, q_h) &= 0 \qquad\quad \forall\, q_h \in P^h, && (3.4b) \end{aligned}$$

where the discrete bilinear forms $a_h(\cdot,\cdot) : [\mathcal{NC}_0^h]^3 \times [\mathcal{NC}_0^h]^3 \to \mathbb{R}$ and $b_h(\cdot,\cdot) : [\mathcal{NC}_0^h]^3 \times P^h \to \mathbb{R}$ are given by

$$a_h(\mathbf{u}, \mathbf{v}) = \nu \sum_{j=1}^{N_Q} (\nabla \mathbf{u}, \nabla \mathbf{v})_j \quad \text{and} \quad b_h(\mathbf{v}, q) = -\sum_{j=1}^{N_Q} (\nabla \cdot \mathbf{v}, q)_j.$$

3.3 The inf-sup condition and the enrichment of velocity space by several global bubble-vector functions

In this section, we will prove that the system (3.4) is a singular problem, *i.e.*, we can not solve the discrete problem (3.4) under the approximated space pair $[\mathcal{NC}_0^h]^3 \times P^h$. From now on, we will limit our analysis to the a family of quasi-regular partitions of $\Omega \subset \mathbb{R}^3$ into disjoint hexahedrons Q_{jkl} of size $h_{x_j} \times h_{y_k} \times h_{z_l}, j = 1, \cdots, N_x, k = 1, \cdots, N_y, l = 1, \cdots, N_z$, and $\overline{\Omega} = \bigcup_{j=1}^{N_x} \bigcup_{k=1}^{N_y} \bigcup_{l=1}^{N_z} \overline{Q}_{jkl}$, where N_x, N_y and N_z are the numbers of hexahedrons in the x-, y- and z- directions, respectively. Also, let $|Q_{jkl}|$ be the volume of hexahedron Q_{jkl}. $|F_{j,k}|$, $|F_{k,l}|$ and $|F_{j,l}|$ are the areas of the faces with $h_{x_j} \times h_{y_k}$, $h_{y_k} \times h_{z_l}$ and $h_{x_j} \times h_{z_l}$, respectively.

3.3.1 The linear system (3.4) is singular

To prove the singularity of the system (3.4), it is sufficient to derive that there exists a non-trivial solution of (3.4) showed that $\mathbf{f} = \mathbf{0}$. Taking $\mathbf{v}_h = \mathbf{u}_h \in [\mathcal{NC}_0^h]^3$ and $q_h = p_h \in P^h$ in (3.4), we know that $b_h(\mathbf{u}_h, p_h) = 0$ and

$a_h(\mathbf{u}_h, \mathbf{u}_h) = 0$ when $\mathbf{f} = \mathbf{0}$. The coercivity of bilinear form $a_h(\cdot, \cdot)$ implies that $\mathbf{u}_h = \mathbf{0}$. Thus, (3.4) reduces to $b_h(\mathbf{v}_h, p_h) = 0 \quad \forall\, \mathbf{v}_h \in [\mathcal{NC}_0^h]^3$. In order to analyze this, set

$$\mathcal{C}^h = \{p_h \in P^h :\ b_h(\mathbf{v}_h, p_h) = 0 \quad \forall\, \mathbf{v}_h \in [\mathcal{NC}_0^h]^3\},$$

for which we will derive that $\dim(\mathcal{C}^h) = N_x^i + N_y^i + N_z^i + 1$, where N_x^i, N_y^i and N_z^i are the numbers of interior nodes in $\mathcal{T}_h$ in the x-, y- and z- directions, respectively. Then we show that the system (3.4) is singular.

Let $p_h \in \mathcal{C}^h$ be arbitrary. Since a element $\mathbf{v}_h \in [\mathcal{NC}_0^h]^3$ has a representation

$$\mathbf{v}_h = \sum_{j=1}^{N_x^i} \sum_{k=1}^{N_y^i} \sum_{l=1}^{N_z^i} \begin{pmatrix} a_{jkl} \\ b_{jkl} \\ c_{jkl} \end{pmatrix} \phi_{jkl}(x, y, z), \qquad \begin{pmatrix} a_{jkl} \\ b_{jkl} \\ c_{jkl} \end{pmatrix} \in \mathbb{R}^3,$$

An elementary calculation leads to

$$\begin{aligned} -b_h(\mathbf{v}_h, p_h) = \sum_{j=1}^{N_x^i} \sum_{k=1}^{N_y^i} \sum_{l=1}^{N_z^i} \Big[& a_{jkl}(p_{j,k,l}|F_{k,l}| - p_{j+1,k,l}|F_{k,l}| + p_{j,k+1,l}|F_{k+1,l}| - p_{j+1,k+1,l}|F_{k+1,l}| \\ & + p_{j,k,l+1}|F_{k,l+1}| - p_{j+1,k,l+1}|F_{k,l+1}| + p_{j,k+1,l+1}|F_{k+1,l+1}| - p_{j+1,k+1,l+1}|F_{k+1,l+1}|) \\ & + b_{jkl}(p_{j,k,l}|F_{j,l}| + p_{j+1,k,l}|F_{j+1,l}| - p_{j,k+1,l}|F_{j,l+1}| - p_{j+1,k+1,l}|F_{j+1,l}| + p_{j,k,l+1}|F_{j,l+1}| \\ & + p_{j+1,k,l+1}|F_{j+1,l+1}| - p_{j,k+1,l+1}|F_{j,l+1}| - p_{j+1,k+1,l+1}|F_{j+1,l+1}|) \\ & + c_{jkl}(p_{j,k,l}|F_{j,k}| + p_{j+1,k,l}|F_{j+1,k}| + p_{j,k+1,l}|F_{j,k+1}| + p_{j+1,k+1,l}|F_{j+1,k+1}| \\ & - p_{j,k,l+1}|F_{j,k}| - p_{j+1,k,l+1}|F_{j+1,k}| - p_{j,k+1,l+1}|F_{j,k+1}| - p_{j+1,k+1,l+1}|F_{j+1,k+1}|) \Big], \end{aligned} \tag{3.5}$$

where $p_{j,k,l}$ denotes the piecewise constant value of p_h on $Q_{jkl} = (x_{j-1}, x_j) \times (y_{k-1}, y_k) \times (z_{l-1}, z_l)$ (See Figure 3.2). Since $b_h(\mathbf{v}_h, p_h) = 0$ for all $\mathbf{v}_h \in [\mathcal{NC}_0^h]^3$, the equation (3.5) implies that

$$
\begin{aligned}
&p_{j,k,l}|F_{k,l}| - p_{j+1,k,l}|F_{k,l}| + p_{j,k+1,l}|F_{k+1,l}| - p_{j+1,k+1,l}|F_{k+1,l}| + p_{j,k,l+1}|F_{k,l+1}| \\
&\qquad - p_{j+1,k,l+1}|F_{k,l+1}| + p_{j,k+1,l+1}|F_{k+1,l+1}| - p_{j+1,k+1,l+1}|F_{k+1,l+1}| = 0, \quad \text{(3.6a)} \\
&p_{j,k,l}|F_{j,l}| + p_{j+1,k,l}|F_{j+1,l}| - p_{j,k+1,l}|F_{j,l+1}| - p_{j+1,k+1,l}|F_{j+1,l}| + p_{j,k,l+1}|F_{j,l+1}| \\
&\qquad + p_{j+1,k,l+1}|F_{j+1,l+1}| - p_{j,k+1,l+1}|F_{j,l+1}| - p_{j+1,k+1,l+1}|F_{j+1,l+1}| = 0, \quad \text{(3.6b)} \\
&p_{j,k,l}|F_{j,k}| + p_{j+1,k,l}|F_{j+1,k}| + p_{j,k+1,l}|F_{j,k+1}| + p_{j+1,k+1,l}|F_{j+1,k+1}| - p_{j,k,l+1}|F_{j,k}| \\
&\qquad - p_{j+1,k,l+1}|F_{j+1,k}| - p_{j,k+1,l+1}|F_{j,k+1}| - p_{j+1,k+1,l+1}|F_{j+1,k+1}| = 0, \quad \text{(3.6c)}
\end{aligned}
$$

for $j = 1, 2, \cdots, N_x^i,\ k = 1, 2, \cdots, N_y^i,\ l = 1, 2, \cdots, N_z^i$. Setting $\alpha_j = h_{x_{j+1}}/h_{x_j}$, $\beta_k = h_{y_{k+1}}/h_{y_k}$ and $\gamma_l = h_{z_{l+1}}/h_{z_l}$, we rewrite the system (3.6) as follows:

$$
\begin{aligned}
&p_{j,k,l} - p_{j+1,k,l} + p_{j,k+1,l}\beta_k - p_{j+1,k+1,l}\beta_k + p_{j,k,l+1}\gamma_l \\
&\qquad - p_{j+1,k,l+1}\gamma_l + p_{j,k+1,l+1}\beta_k\gamma_l - p_{j+1,k+1,l+1}\beta_k\gamma_l = 0, \quad \text{(3.7a)} \\
&p_{j,k,l} + p_{j+1,k,l}\alpha_j - p_{j,k+1,l}\gamma_l - p_{j+1,k+1,l}\alpha_j + p_{j,k,l+1}\gamma_l \\
&\qquad + p_{j+1,k,l+1}\alpha_j\gamma_l - p_{j,k+1,l+1}\gamma_l - p_{j+1,k+1,l+1}\alpha_j\gamma_l = 0, \quad \text{(3.7b)} \\
&p_{j,k,l} + p_{j+1,k,l}\alpha_j + p_{j,k+1,l}\beta_k + p_{j+1,k+1,l}\alpha_j\beta_k - p_{j,k,l+1} \\
&\qquad - p_{j+1,k,l+1}\alpha_j - p_{j,k+1,l+1}\beta_k - p_{j+1,k+1,l+1}\alpha_j\beta_k = 0, \quad \text{(3.7c)}
\end{aligned}
$$

for $j = 1, 2, \cdots, N_x^i,\ k = 1, 2, \cdots, N_y^i$ and $l = 1, 2, \cdots, N_z^i$. It is apparent that the system (3.7) is of rank 3, that is, any three values of the eight $p_{j',k',l'},\ j' = j, j+1,\ k' = k, k+1$ and $l' = l, l+1$, can be calculated in terms of the other five values.

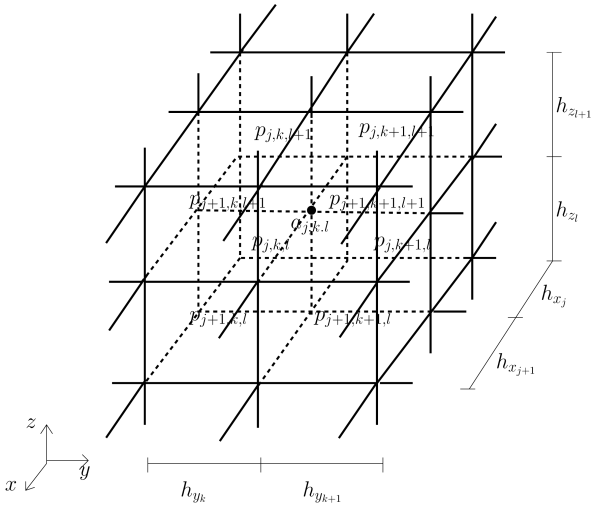


Figure 3.2: a_{jkl}, b_{jkl} and c_{jkl} denote the coefficients of the P_1-nonconforming element at the vertex (x_j, y_k, z_l) for the approximated velocity field. $p_{j,k,l}$ means the piecewise constant value as approximation of pressure on $[x_{j-1}, x_j] \times [y_{k-1}, y_k] \times [z_{l-1}, z_l]$.

Without loss of generality, from (3.7), we obtain

$$\begin{pmatrix} -1 & -\beta_k & -\gamma_l \\ \alpha_j & -\alpha_j & \alpha_j\gamma_l \\ \alpha_j & \alpha_j\beta_k & -\alpha_j \end{pmatrix} \begin{pmatrix} p_{j+1,k,l} \\ p_{j+1,k+1,l} \\ p_{j+1,k,l+1} \end{pmatrix} = \begin{pmatrix} -1 & -\beta_k & -\gamma_l & -\beta_k\gamma_l & \beta_k\gamma_l \\ -1 & \gamma_l & -\gamma_l & \gamma_l & \alpha_j\gamma_l \\ -1 & -\beta_k & 1 & \beta_k & \alpha_j\beta_k \end{pmatrix} \begin{pmatrix} p_{j,k,l} \\ p_{j,k+1,l} \\ p_{j,k,l+1} \\ p_{j,k+1,l+1} \\ p_{j+1,k+1,l+1} \end{pmatrix}. \tag{3.8}$$

The formula (3.8) can be applied subsequently to write $[p_{j',k,l} \;\; p_{j',k+1,l} \;\; p_{j',k,l+1}]^T$ for $j' = j+2, \cdots, N_x$. The same procedure of course applies backwards to represent $[p_{j',k,l} \;\; p_{j',k+1,l} \;\; p_{j',k,l+1}]^T$ for $j' = 1, \cdots, j-1$. In each procedure, we use $p_{j',k+1,l+1}$ as one new degree of freedom for $j' = 1, \cdots, j-1, j+2, \cdots, N_x$. In these procedures, we get $(N_x^i - 1)$ additional degrees of freedom. Thus, for the two strips, all the pressure values $[p_{j',k,l} \;\; p_{j',k+1,l} \;\; p_{j',k,l+1}]^T$, $j' = 1, \cdots, N_x$, are represented in terms of $5 + (N_x^i - 1)$ pressure values.

Using these pressure values, $[p_{j',k,l} \quad p_{j',k+1,l} \quad p_{j',k,l+1} \quad p_{j',k+1,l+1}]^T$ for $j' = 1, \cdots, N_x^i$, and additional $(N_y^i - 1)$ pressure values, we will find the values for $[p_{j+1,k',l} \;\; p_{j,k',l} \;\; p_{j+1,k',l+1}]^T$, $k' = 1, \cdots, k-1, k+2, \cdots, N_y$. First, let $[p_{j,k+1,l} \;\; p_{j+1,k+1,l} \;\; p_{j,k+1,l+1} \;\; p_{j+1,k+1,l+1}]^T$ be the determined four pressure values by previous step. And then we get the values $[p_{j+1,k',l} \;\; p_{j,k',l} \;\; p_{j+1,k',l+1}]^T$, $k' = k+2, \cdots, N_y$, with pressure value $p_{j,k',l+1}, k' = k+2, \cdots, N_y$, as follow.

$$\begin{pmatrix} -\beta_k & \beta_k & -\beta_k\gamma_l \\ -\alpha_j & -\gamma_l & -\alpha_j\gamma_l \\ \alpha_j\beta_k & \beta_k & -\alpha_j\beta_k \end{pmatrix} \begin{pmatrix} p_{j+1,k',l} \\ p_{j,k',l} \\ p_{j+1,k',l+1} \end{pmatrix} = \begin{pmatrix} -1 & 1 & -\gamma_l & \gamma_l & -\beta_k\gamma_l \\ -1 & -\alpha_j & -\gamma_l & -\alpha_j\gamma_l & \gamma_l \\ -1 & -\alpha_j & 1 & \alpha_j & \beta_k \end{pmatrix} \begin{pmatrix} p_{j,k'-1,l} \\ p_{j+1,k'-1,l} \\ p_{j,k'-1,l+1} \\ p_{j+1,k'-1,l+1} \\ p_{j,k',l+1} \end{pmatrix}. \tag{3.9}$$

The same procedure of course applies backwards to represent $[p_{j+1,k',l} \;\; p_{j,k',l} \;\; p_{j+1,k',l+1}]^T$, $k' = 1, \cdots, k-1$, by aid of pressure values $p_{j,k',l+1}, k' = 1, \cdots, k-1$. Other pressure values $p_{j',k',l'}$ for $j' = 1, \cdots, j-1, j+2, \cdots, N_x$, $k' = 1, \cdots, k-1, k+2, \cdots, N_y$ and $l' = l, l+1$ are determined by (3.7) uniquely.

Next, by using the quite similar procedures, we will find the values for $[p_{j,k,l'} \;\; p_{j+1,k,l'} \;\; p_{j+1,k+1,l'}]^T$, $l' = 1, \cdots, l-1, l+2, \cdots, N_z$, by aid of $(N_z^i - 1)$

pressure values, for instance $p_{j,k+1,l'}, l' = 1, \cdots, l-1, l+2, \cdots, N_z$, as follow.

$$\begin{pmatrix} \gamma_l & -\gamma_l & -\beta_k\gamma_l \\ \gamma_l & \alpha_j\gamma_l & -\alpha_j\gamma_l \\ -1 & -\alpha_j & -\alpha_j\beta_k \end{pmatrix} \begin{pmatrix} p_{j,k,l'} \\ p_{j+1,k,l'} \\ p_{j+1,k+1,l'} \end{pmatrix} = \begin{pmatrix} -1 & 1 & -\beta_k & \beta_k & -\beta_k\gamma_l \\ -1 & -\alpha_j & \gamma_l & \alpha_j & \gamma_l \\ -1 & -\alpha_j & -\beta_k & -\alpha_j\beta_k & \beta_k \end{pmatrix} \begin{pmatrix} p_{j,k,l'-1} \\ p_{j+1,k,l'-1} \\ p_{j,k+1,l'-1} \\ p_{j+1,k+1,l'-1} \\ p_{j,k+1,l'} \end{pmatrix}. \tag{3.10}$$

The formula (3.10) can be applied subsequently to writing $[p_{j,k,l'} \;\; p_{j+1,k,l'} \;\; p_{j+1,k+1,l'}]^T$ for $l' = l+2, \cdots, N_z$. The same procedure of course applies backwards to represent $[p_{j,k,l'} \;\; p_{j+1,k,l'} \;\; p_{j+1,k+1,l'}]^T$ for $l' = 1, \cdots, l-1$. Thus, using above three procedures, all the pressure values in this problem are determined by $(N_x^i + N_y^i + N_z^i + 2)$ pressure values.

Remark 3.3.1. *There arises a commutativity issue and we show it as follows: For instance, if* $p_{j',k',l'}$, $j' = j, j+1$, $k' = k, k+1$, $l' = l, l+1$, *are given, there are two ways of finding* $p_{j+2,k+2,l'}$, $l' = l, l+1$*: first, use* $p_{j+1,k',l'}$ *and* $p_{j+2,k+1,l+1}$ *to find* $[p_{j+2,k,l} \;\; p_{j+2,k+1,l} \;\; p_{j+2,k,l+1}]^T$ *and then use* $p_{j',k+1,l'}$ *and* $p_{j+1,k+2,l+1}$ *to find* $[p_{j+2,k+2,l} \;\; p_{j+2,k+2,l+1} \;\; p_{j+1,k+2,l}]^T$, $j' = j+1, j+2$, $k' = k, k+1$, $l' = l, l+1$*; second, use* $p_{j',k+1,l'}$ *and* $p_{j+1,k+2,l+1}$ *to find* $[p_{j,k+2,l} \;\; p_{j+1,k+2,l} \;\; p_{j,k+2,l+1}]^T$ *and then use* $p_{j+1,k',l'}$ *and* $p_{j+2,k+1,l+1}$ *to find* $[p_{j+2,k+1,l} \;\; p_{j+2,k+2,l} \;\; p_{j+2,k+2,l+1}]^T$, $j' = j, j+1$, $k' = k+1, k+2$, $l' = l, l+1$. *The pressure values* $p_{j+2,k+2,l'}$, $l' = l, l+1$, *determined by two ways have to be identical.*

After the all $p_{j,k,l}$ are expressed in terms of any $(N_x^i + N_y^i + N_z^i + 2)$ values, say $[p_{j_0,k_0,l_0} \;\; p_{j_0,k_0+1,l_0}$
$p_{j_0,k_0,l_0+1} \;\; p_{j_0,k_0+1,l_0+1} \;\; p_{j_0+1,k_0+1,l_0+1}]^T$ and $[p_{j',k_0+1,l_0+1} \;\; p_{j_0,k',l_0+1} \;\; p_{j_0,k_0+1,l'}]^T$, where $j' = 1, \cdots, j_0-1, j_0+2, \cdots, N_x$, $k' = 1, \cdots, k_0-1, k_0+2, \cdots, N_y$ and

$l' = 1, \cdots, l_0 - 1, l_0 + 2, \cdots, N_z$. The mean value zero constraint $\int_\Omega p_h \, d\mathbf{V} = 0$ should be fulfilled. A direct calculation shows the following relationships:

$$\int_\Omega p_h \, d\mathbf{V} = \begin{cases} \dfrac{|\Omega|}{|F|} \displaystyle\sum_{m_1=0}^{1} \Big(\sum_{m_2=0}^{1} p_{j_0,k_0+m_1,l_0+m_2} |F_{k_0+m_1,l_0+m_2}| \Big) & \text{if } N_x \text{ or } N_y \text{ are even}, \\ \dfrac{1}{|F|} \displaystyle\sum_{m_1=0}^{1} \Big(\sum_{m_2=0}^{1} (|\Omega| + (-1)^{1-m_2} |Q_{j_0,k_0+m_1,l_0+1-m_2}|) \\ \qquad\qquad p_{j_0,k_0+m_1,l_0+m_2} |F_{k_0+m_1,l_0+m_2}| \Big) & \text{otherwise}, \end{cases} \tag{3.11}$$

where $|\Omega|$ is the volume of the domain Ω and $|F| := |F_{k_0,l_0}| + |F_{k_0,l_0+1}| + |F_{k_0+1,l_0}| + |F_{k_0+1,l_0+1}|$.

It is clear that the constraint (3.11) restricts the values of p_{j_0,k_0,l_0}, p_{j_0,k_0+1,l_0}, p_{j_0,k_0,l_0+1} and p_{j_0,k_0+1,l_0+1} to be dependent on each other value, which implies that $\dim(\mathcal{C}^h) = N_x^i + N_y^i + N_z^i + 1$. In particular, at least one of the four values must have opposite sign. The similar argument also applies to the dependency and signs of any four pressure values which have two common faces. Thus, we conclude that p_h has a non-trivial solution and of $(N_x^i + N_y^i + N_z^i + 1)$ dimension.

Summarizing the above, we obtain the following theorem.

Theorem 3.3.2. *The dimension of $\mathcal{C}^h$ is $N_x^i + N_y^i + N_z^i + 1$.*

An immediate consequence of the result is as follows.

Corollary 3.3.3. *There exists non-unique discrete solution $(\mathbf{u}_h, p_h) \in [\mathcal{NC}_0^h]^3 \times P^h$ of (3.4).*

By Theorem 3.3.2, we notice that the discrete Stokes problem can not be solved by using the only P_1-nonconforming element even on hexahedral meshes in general. To resolve this difficulty, we have to add $(N_x^i + N_y^i + N_z^i + 1)$ bubble functions to the P_1-nonconforming element by using another hexahedral nonconforming finite elements. Then, the discrete problem of the

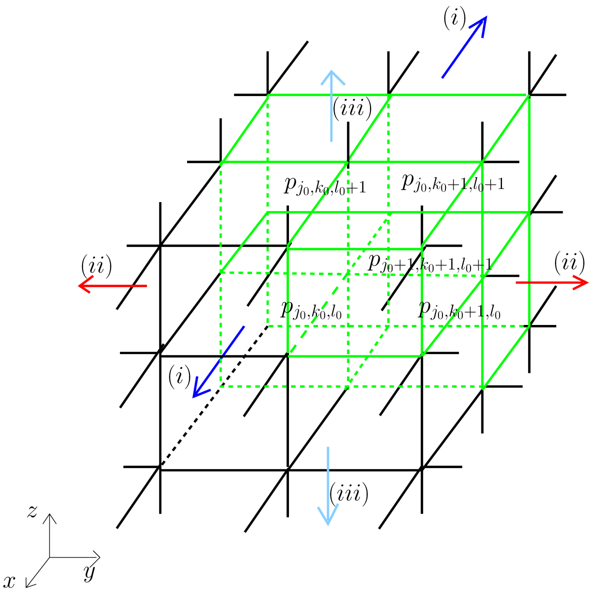


Figure 3.3: From the values of p_{j_0,k_0,l_0}, p_{j_0,k_0+1,l_0}, p_{j_0,k_0,l_0+1} , p_{j_0,k_0+1,l_0+1}, p_{j_0+1,k_0+1,l_0+1} and additional $(N_x^i + N_y^i + N_z^i - 3)$ pressure values we construct $p_{j,k,l}$ for all j, k, l with ordering $(i) \to (ii) \to (iii)$.

Stokes equations is solvable in the modified nonconforming element space and one proves its solvability in Theorem 3.3.5.

3.3.2 Enrichment by $(N_x^i + N_y^i + N_z^i + 1)$ global bubble-vectors

In order to ensure the solvability, we add $(N_x^i+N_y^i+N_z^i+1)$ global bubble-vectors to the nonconforming space $[\mathcal{NC}_0^h]^3$. It is different from the two dimensional case [38]. In [38], since the dimension of $\mathcal{C}^h$ is one, we need only one bubble-vector function. However, we should consider $(N_x^i + N_y^i + N_z^i + 1)$ bubble-vector functions in three dimensional case.

The $DSSY$ nonconforming elements

On a reference domain $\widehat{R} := [-1,1]^3$, the $DSSY$ nonconforming element space is defined by

$$\mathcal{S}^*(\widehat{R}) = \mathrm{Span}\{1, x, y, z, \theta(x) - \theta(y), \theta(x) - \theta(z)\},$$

where $\theta(s) = s^2 - \frac{5}{3}s^4$ and the basis functions $\psi_j : \widehat{R} \to \mathbb{R}$, $j = 1, \cdots, 6$, are given by

$$\psi_j(M_k) = \frac{1}{|F_k|}\int_{F_k} \psi_j d\mathbf{x} = \delta_{jk} \quad \forall k = 1, \cdots, 6.$$

Then one sees that

$$\mathcal{S}^*(\widehat{R}) = \mathrm{Span}\{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6\}$$

and the degrees of freedom associated with $\mathcal{S}^*(R)$ are the function values at the six face-centroids of $\widehat{R}$, or equivalently the integrals of function on the six faces.

The enrichment of velocity space

We will enrich the space $[\mathcal{NC}_0^h]^3$ by adding $(N_x^i + N_y^i + N_z^i + 1)$ global bubble-vectors from the $DSSY$ nonconforming finite element space, in order to approximate the velocity field.

For the hexahedral type, choose any interior face $F_{k_1,k_2} = \partial Q_{k_1} \bigcap \partial Q_{k_2}$ with its face-centroid M_{k_1,k_2} from $\mathcal{T}_h$. On this face, a basis function $\boldsymbol{\psi} = \boldsymbol{\psi}(\mathbf{x})$ is defined by $\boldsymbol{\psi}|_{Q_{k_j}} \in [\mathcal{S}^*(Q_{k_j})]^3$ for $j = 1, 2$,

$$
\begin{aligned}
\frac{1}{|F_{k_1,k_2}|} \int_{F_{k_1,k_2}} \boldsymbol{\psi}\, d\mathbf{x} &= \boldsymbol{\nu}_{M_{k_1,k_2}}, \text{ which is the unit normal vector from } Q_{k_1} \text{ to } Q_{k_2}, \\
\text{and} \int_F \boldsymbol{\psi}\, d\mathbf{x} &= \mathbf{0} \text{ for all other faces } F \text{ of } (\partial Q_{k_1} \bigcup \partial Q_{k_2}) \setminus F_{k_1,k_2}.
\end{aligned} \tag{3.12}
$$

Extend by zero to all $Q_\ell,\ \ell \neq k_1, k_2$ in $\mathcal{T}_h$, still denoted by $\boldsymbol{\psi}$. Then, we now introduce the nonconforming finite dimensional space $\mathbf{V}^h$ as following

$$
\mathbf{V}^h = [\mathcal{NC}_0^h]^3 \ \oplus\ \mathbf{B}^h,
$$

where $\mathbf{B}^h = \text{Span}\{\boldsymbol{\psi}^j(\mathbf{x}), j = 1, \cdots, (N_x^i + N_y^i + N_z^i + 1)\}$ is the bubble-vector space. Observe that

$$
(\nabla \cdot \boldsymbol{\psi}, 1)_{0,Q_{k_1}} = |F_{k_1,k_2}| = -(\nabla \cdot \boldsymbol{\psi}, 1)_{0,Q_{k_2}}. \tag{3.13}
$$

To emphasize the use of modified space $\mathbf{V}^h$, we rewrite the discrete weak formulation of (3.2): Find a pair $(\mathbf{u}_h, p_h) \in \mathbf{V}^h \times P^h$ such that

$$
a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall\ \mathbf{v}_h \in \mathbf{V}^h, \tag{3.14a}
$$

$$
b_h(\mathbf{u}_h, q_h) = 0 \qquad \forall\ q_h \in P^h. \tag{3.14b}
$$

We will show that $\mathbf{V}^h \times P^h$ is a stable space pair for solving (3.14). First, we will investigate how to take $(N_x^i + N_y^i + N_z^i + 1)$ bubble-vector functions.

Location of bubble-vector functions

By using the finite space pair $[\mathcal{NC}_0^h]^3 \times P^h$, we have to overcome singularity of the discrete system. As the key idea, we consider bubble-vector functions by using $DSSY$ nonconforming finite element in three dimensions. In two dimensional case [38], location of bubble-vector function is not an important condition because the problem is solvable by adding only a one-dimensional additional bubble-vector functions no matter which rectangle their supports are. This is no longer valid in the three dimensional case.

Without loss of generality, we assume that three bubble-vector functions $\boldsymbol{\psi}^j,\ j = 1, 2, 3$, are located two adjacent hexahedrons $Q_{j_0,k_0,l_0} \cup Q_{j_0,k_0+1,l_0},\ Q_{j_0,k_0+1,l_0} \cup Q_{j_0,k_0+1,l_0+1}$ and $Q_{j_0,k_0,l_0+1} \cup Q_{j_0,k_0+1,l_0+1}$, respectively. If we calculate $b_h(\boldsymbol{\psi}^j, p_h) = 0, j = 1, 2, 3$, then we get

$$
\begin{aligned}
&(p_{j_0,k_0,l_0} - p_{j_0,k_0+1,l_0})|F_{j_0,l_0}| = 0, \\
&(p_{j_0,k_0+1,l_0} - p_{j_0,k_0+1,l_0+1})|F_{j_0,k_0+1}| = 0, \\
&(p_{j_0,k_0,l_0+1} - p_{j_0,k_0+1,l_0+1})|F_{j_0,l_0+1}| = 0.
\end{aligned} \tag{3.15}
$$

Also, we recall the following relationship (3.11)

$$
c_1 p_{j_0,k_0,l_0} + c_2 p_{j_0,k_0+1,l_0} + c_3 p_{j_0,k_0,l_0+1} + c_4 p_{j_0,k_0+1,l_0+1} = 0, \tag{3.16}
$$

where $c_1, \cdots, c_4$ are positive constant depending on mesh size. Since the signs of $p_{j_0,k',l'},\ k' = k_0, k_0 + 1,\ l' = l_0, l_0 + 1$, are same from the above equations, possible solution of (3.15) must be zero. Next, we assume that N_x^i bubble-vector functions $\boldsymbol{\psi}^j$ are located two adjacent hexahedron $Q_{j,k_0+1,l_0+1} \cup$

$Q_{j+1,k_0+1,l_0+1},\ j = 1, \cdots, N_x^i$. Then we have

$$(p_{j,k_0+1,l_0+1} - p_{j+1,k_0+1,l_0+1})|F_{k_0+1,l_0+1}| = 0.$$

Since p_{j_0,k_0+1,l_0+1} is zero from (3.15)–(3.16), p_{j_0+1,k_0+1,l_0+1} is also zero. Similar argument can be applied sequentially to all $j \neq j_0$. Thus, all p_{j,k_0+1,l_0+1} are zero. Then, by (3.8), we conclude that $[p_{j,k_0,l_0} \ \ p_{j,k_0+1,l_0} \ \ p_{j,k_0,l_0+1}]^T$ are zero for all $j \neq j_0$. In addition, we assume that $(N_y^i - 1)$ bubble-vector functions $\boldsymbol{\psi}^k$ are located two adjacent hexahedron $Q_{j_0,k,l_0+1} \cup Q_{j_0,k+1,l_0+1},\ k = 1, \cdots, k_0 - 1, k_0 + 1, \cdots, N_y^i$. Then we obtain

$$(p_{j_0,k,l_0+1} - p_{j_0,k+1,l_0+1})|F_{j_0,l_0+1}| = 0.$$

By the similar argument, all p_{j_0,k,l_0+1} are zero. By (3.9), we know that $[p_{j_0+1,k,l_0} \ \ p_{j_0,k,l_0} \ \ p_{j_0+1,k,l_0+1}]^T$ are zero for all k . Finally, we assume that $(N_z^i - 1)$ bubble-vectors $\boldsymbol{\psi}^l$ are located two adjacent hexahedrons $Q_{j_0,k_0+1,l} \cup Q_{j_0,k_0+1,l+1},\ l = 1, \cdots, l_0 - 1, l_0 + 1, \cdots, N_z^i$.
Then we obtain

$$(p_{j_0,k_0+1,l} - p_{j_0,k_0+1,l+1})|F_{j_0,k_0}| = 0.$$

By the similar argument, all values of $p_{j_0,k_0+1,l}$ are zero. And by (3.10) we conclude that $[p_{j_0,k_0,l} \ \ p_{j_0+1,k_0,l} \ \ p_{j_0+1,k_0+1,l}]^T$ are zero for all l. The values $p_{j,k,l},\ j \neq j_0, j_0 + 1,\ k \neq k_0, k_0 + 1,\ l \neq l_0, l_0 + 1$, can be determined to zero from (3.8)–(3.10). Hence, the problem (3.14) with $(N_x^i + N_y^i + N_z^i + 1)$ bubble-vector functions has the trivial solution, provided that $\mathbf{f} \equiv \mathbf{0}$.

3.3.3 The inf-sup condition

We notice that the following result is very much meaningful.

Lemma 3.3.4. *We have*

$$\sup_{\mathbf{v}\in\mathbf{V}^h} b_h(\mathbf{v},p) > 0 \quad \forall p \in P^h\backslash\{0\}. \tag{3.17}$$

The proof of Lemma 3.3.4 is similar to that for Lemma 3.4 in [38]. The details are omitted.

As usual, let $|\cdot|_{1,h}$ denote the (broken) energy semi-norm by $|\mathbf{v}|_{1,h} = \sqrt{\nu^{-1}a_h(\mathbf{v},\mathbf{v})}$, which is a norm over $\mathbf{V}^h$ due to Poincaré lemma. Also, denote by $\|\cdot\|_{m,h}$ and $|\cdot|_{m,h}$ the usual mesh-dependent norm and semi-norm:

$$\|\mathbf{v}\|_{m,h} = \left[\sum_{Q\in\mathcal{T}_h} \|\mathbf{v}\|_{H^m(Q)}\right]^{1/2} \quad \text{and} \quad |\mathbf{v}|_{m,h} = \left[\sum_{Q\in\mathcal{T}_h} |\mathbf{v}|_{H^m(Q)}\right]^{1/2},$$

respectively.

We are now ready to state and prove the following stability result.

Theorem 3.3.5 (The weak discrete inf-sup condition)**.** *Let $(\mathcal{T}_h)_{0<h<1}$ be a family of triangulations of Ω into hexahedrons. Suppose that the quasi-regular property of $\mathcal{T}_h$ holds, i.e. there is a constant $\rho > 0$ such that $h/h_{\min} \leq \rho$, where $h := \max_{j,k,l}\{h_{x_j}, h_{y_k}, h_{z_l}\}$ and $h_{\min} := \min_{j,k,l}\{h_{x_j}, h_{y_k}, h_{z_l}\}$. Then there exists a bound $\beta(h) > 0$, dependent on the mesh size h, such that, for all sufficiently small h,*

$$\sup_{\mathbf{v}_h\in\mathbf{V}^h} \frac{b_h(\mathbf{v}_h, q_h)}{|\mathbf{v}_h|_{1,h}} \geq \beta(h)\|q_h\|_{0,h} \quad \forall q_h \in P^h. \tag{3.18}$$

Proof. If $q_h = 0$, the inequality (3.18) is trivial. Thus it is sufficient to consider the case of $q_h \in P^h$ with $q_h \not\equiv 0$. We recall that the bubble-vector functions

$\boldsymbol{\psi}_{x,j}, \boldsymbol{\psi}_{y,k}, \boldsymbol{\psi}_{z,l}$ and $\boldsymbol{\psi}_{N_B} \in \mathbf{B}^h$ are defined by (3.12) have the supports in the manner described in §3.3.2, where N_B is the dimension of $\mathcal{C}^h$, *i.e.*, $N_B = N_x^i + N_y^i + N_z^i + 1$. We will construct $\mathbf{v}_h \in \mathbf{V}^h$ satisfying $b_h(\mathbf{v}_h, q_h) \geq \beta |\mathbf{v}_h|_{1,h} \|q_h\|_{0,h}$ for some constant $\beta = \beta(h) > 0$. Without loss of generality, we assume that the supports of bubble-vector functions $\boldsymbol{\psi}_{x,j'}, \boldsymbol{\psi}_{y,k'}, \boldsymbol{\psi}_{z,l'}$ and $\boldsymbol{\psi}_{N_B}$ are $Q_{j',k_0+1,l_0+1} \cup Q_{j'+1,k_0+1,l_0+1},\ j' = 1, \cdots, N_x^i$, $Q_{j_0,k',l_0+1} \cup Q_{j_0,k'+1,l_0+1},\ k' = 1, \cdots, N_y^i$, $Q_{j_0,k_0+1,l'} \cup Q_{j_0,k_0+1,l'+1},\ l' = 1, \cdots, N_z^i$, and $Q_{j_0,k_0,l_0} \cup Q_{j_0,k_0+1,l_0}$, respectively.

Let us take

$$\mathbf{v}_h = \sum_{j=1}^{N_x^i}\sum_{k=1}^{N_y^i}\sum_{l=1}^{N_z^i} \begin{pmatrix} a_{jkl} \\ b_{jkl} \\ c_{jkl} \end{pmatrix} \varphi_{jkl} + \sum_{j=1}^{N_x^i} d_j \boldsymbol{\psi}_{x,j} + \sum_{k=1}^{N_y^i} e_k \boldsymbol{\psi}_{y,k} + \sum_{l=1}^{N_z^i} f_l \boldsymbol{\psi}_{z,l} + g\boldsymbol{\psi}_{N_B}, \tag{3.19}$$

where

$$\begin{aligned}
a_{jkl} &= q_{j,k,l}|F_{k,l}| - q_{j+1,k,l}|F_{k,l}| + q_{j,k+1,l}|F_{k+1,l}| - q_{j+1,k+1,l}|F_{k+1,l}| + q_{j,k,l+1}|F_{k,l+1}| \\
&- q_{j+1,k,l+1}|F_{k,l+1}| + q_{j,k+1,l+1}|F_{k+1,l+1}| - q_{j+1,k+1,l+1}|F_{k+1,l+1}|, && \text{(3.20a)}\\
b_{jkl} &= q_{j,k,l}|F_{j,l}| + q_{j+1,k,l}|F_{j+1,l}| - q_{j,k+1,l}|F_{j,l}| - q_{j+1,k+1,l}|F_{j+1,l}| + q_{j,k,l+1}|F_{j,l+1}| \\
&+ q_{j+1,k,l+1}|F_{j+1,l+1}| - q_{j,k+1,l+1}|F_{j,l+1}| - q_{j+1,k+1,l+1}|F_{j+1,l+1}|, && \text{(3.20b)}\\
c_{jkl} &= q_{j,k,l}|F_{j,k}| + q_{j+1,k,l}|F_{j+1,k}| + q_{j,k+1,l}|F_{j,k+1}| + q_{j+1,k+1,l}|F_{j+1,k+1}| - q_{j,k,l+1}|F_{j,k}| \\
&- q_{j+1,k,l+1}|F_{j+1,k}| - q_{j,k+1,l+1}|F_{j,k+1}| - q_{j+1,k+1,l+1}|F_{j+1,k+1}|, && \text{(3.20c)}\\
d_j &= (q_{j,k_0+1,l_0+1} - q_{j+1,k_0+1,l_0+1})|F_{k_0+1,l_0+1}|, && \text{(3.20d)}\\
e_k &= (q_{j_0,k,l_0+1} - q_{j_0,k+1,l_0+1})|F_{j_0,l_0+1}|, && \text{(3.20e)}\\
f_l &= (q_{j_0,k_0+1,l} - q_{j_0,k_0+1,l+1})|F_{j_0,k_0}|, && \text{(3.20f)}\\
g &= (q_{j_0,k_0,l_0} - q_{j_0,k_0+1,l_0})|F_{j_0,l_0}|, && \text{(3.20g)}
\end{aligned}$$

and φ_{jkl} is the P_1-nonconforming hexahedral basis function for $\mathcal{NC}_0^h(\Omega)$, based on the vertex (x_j, y_k, z_l).

Since $\int_\Omega q_h \, d\mathbf{V} = \sum_{j=1}^{N_x} \sum_{k=1}^{N_y} \sum_{l=1}^{N_z} q_{j,k,l} h_{x_j} h_{y_k} h_{z_l} = 0$, the value q_{j_0,k_0,l_0} can be expressed as the linear combination of the other values of $q_{j,k,l}$'s. Denote by $\underset{\sim}{a}$ the $(3N_x^i N_y^i N_z^i + N_B)$-dimensional vector whose components consist of the coefficients $a_{jkl},\ b_{jkl},\ c_{jkl},\ d_j,\ e_k,\ f_l$ and g for all $j = 1, \cdots, N_x^i,\ k = 1, \cdots, N_y^i$ and $l = 1, \cdots, N_z^i$. Also, designate by $\underset{\sim}{q}$ the $(N_x N_y N_z - 1)$-dimensional vector whose components consist of all $q_{j,k,l}$'s except q_{j_0,k_0,l_0}. Then the relations of (3.20) can be put in matrix form as follow:

$$\underset{\sim}{a} = \mathbb{H}\underset{\sim}{q},$$

where $\mathbb{H}$ denote the $(3N_x^i N_y^i N_z^i + N_B) \times (N_x N_y N_z - 1)$ matrix whose entries contain only the information on hexahedral sizes h_{x_j}'s, h_{y_k}'s and h_{z_l}'s. Obviously we have $\|\underset{\sim}{a}\|^2 = \underset{\sim}{q}^T \mathbb{H}^T \mathbb{H} \underset{\sim}{q} \geq 0$, where $\|\cdot\|$ denotes the Euclidean norm. Indeed, if $\|\underset{\sim}{a}\|^2 = 0$, then $a_{jkl} = b_{jkl} = c_{jkl} = d_j = e_k = f_l = 0 \ \forall j, k, l$ and $g = 0$. However, we want to claim that $\|\underset{\sim}{a}\|^2 > 0$ whenever $q_h \not\equiv 0$. As seen in the proof of Lemma 3.3.4, both $a_{jkl} = b_{jkl} = c_{jkl} = 0$ and $d_j = e_k = f_l = g = 0$ cannot be hold simultaneously. This contradiction proves that $\|\underset{\sim}{a}\|^2 > 0$. Consequently, the $(N_x N_y N_z - 1) \times (N_x N_y N_z - 1)$ matrix $\mathbb{H}^T \mathbb{H}$ is a symmetric positive-definite matrix, and therefore, we get

$$\underset{\sim}{z}^T \mathbb{H}^T \mathbb{H} \underset{\sim}{z} \geq \lambda_1 \|\underset{\sim}{z}\|^2 \quad \forall \underset{\sim}{z} \in \mathbb{R}^{N_x N_y N_z - 1},$$

where $\lambda_1 = \lambda_1(h) > 0$ is a positive minimum eigenvalue of $\mathbb{H}^T \mathbb{H}$. Then we obtain that

$$\|\underset{\sim}{a}\| \geq \lambda_1^{1/2} \|\underset{\sim}{q}\|. \tag{3.21}$$

Finally, we will show that

$$\|\underset{\sim}{a}\|^2 \geq C_1^{-1} |\mathbf{v}_h|_{1,h}^2, \tag{3.22a}$$

$$\|\underset{\sim}{a}\|^2 \geq C_2^{-1} \|q_h\|_{0,h}^2, \tag{3.22b}$$

where both C_1 and C_2 are positive constant depending on h. Let $S_1 := \{(j_0, k_0, l_0)\}$ and $S_2 := S_1 \cup \{(j', k_0+1, l_0+1)\} \cup \{(j_0, k', l_0+1)\} \cup \{(j_0, k_0+1, l')\}$ for $j' = 1, \cdots, N_x,\ k' = 1, \cdots, N_y,\ l' = 1, \cdots, N_z$. To show (3.22), we represent $\mathbf{v}_h$ of (3.19) by the following form based not on vertex indices but on element indices as follows:

For $\mathbf{x} \in Q_{jkl}$,

$$-\mathbf{v}_h(\mathbf{x}) = \begin{cases} \displaystyle\sum_{m=1}^{N_{jkl}} \begin{pmatrix} \tilde{a}_{jkl}^m \\ \tilde{b}_{jkl}^m \\ \tilde{c}_{jkl}^m \end{pmatrix} \varphi_{jkl}^m(\mathbf{x}) + \sum_{m=1}^{N_{jkl}^B} \Big(\tilde{d}_j^m \boldsymbol{\psi}_{x,j}^m(\mathbf{x}) + \tilde{e}_k^m \boldsymbol{\psi}_{y,k}^m(\mathbf{x}) + \tilde{f}_l^m \boldsymbol{\psi}_{z,l}^m + \tilde{g}^m \boldsymbol{\psi}_{N_B}^m(\mathbf{x}) \Big) & \text{if } (j,k,l) \in S_2, \\ \displaystyle\sum_{m=1}^{N_{jkl}} \begin{pmatrix} \tilde{a}_{jkl}^m \\ \tilde{b}_{jkl}^m \\ \tilde{c}_{jkl}^m \end{pmatrix} \varphi_{jkl}^m(\mathbf{x}) & \text{otherwise,} \end{cases}$$

where N_{jkl} and N_{jkl}^B denote the numbers of basis functions of $\mathcal{NC}_0^h(\Omega)$ and $\mathbf{B}^h$ with a support Q_{jkl}, respectively. Here, we note that $(\tilde{a}_{jkl}^m, \tilde{b}_{jkl}^m, \tilde{c}_{jkl}^m, \tilde{d}_j^m, \tilde{e}_k^m, \tilde{f}_l^m, \tilde{g}^m)$ is the one of $\{(a_{jkl}, b_{jkl}, c_{jkl}, d_j, e_k, f_l, g)\}$ and $(\varphi_{jkl}^m, \boldsymbol{\psi}_{x,j}^m, \boldsymbol{\psi}_{y,k}^m, \boldsymbol{\psi}_{z,l}^m, \boldsymbol{\psi}_{N_B}^m)$ is the one of $\{(\varphi_{jkl}, \boldsymbol{\psi}_{x,j}, \boldsymbol{\psi}_{y,k}, \boldsymbol{\psi}_{z,l}, \boldsymbol{\psi}_{N_B})\}$ given in (3.19).

Using the inequality $|\sum_{l=1}^{N} a_l|^2 \leq N \sum_{l=1}^{N} |a_l|^2$ for $a_l \in \mathbb{R}$, we have

$$
\begin{aligned}
|\mathbf{v}_h|_{1,h}^2 = & \sum_{(j,k,l)\in S\setminus S_2} \int_{Q_{jkl}} \Big| \sum_{m=1}^{N_{jkl}} \tilde{a}_{jkl}^m \nabla\varphi_{jkl}^m \Big|^2 + \Big| \sum_{m=1}^{N_{jkl}} \tilde{b}_{jkl}^m \nabla\varphi_{jkl}^m \Big|^2 + \Big| \sum_{m=1}^{N_{jkl}} \tilde{c}_{jkl}^m \nabla\varphi_{jkl}^m \Big|^2 d\mathbf{V} \\
& + \sum_{(j,k,l)\in S_2} \int_{Q_{jkl}} \Big| \sum_{m=1}^{N_{jkl}} \tilde{a}_{jkl}^m \nabla\varphi_{jkl}^m + \sum_{m=1}^{N_{jkl}^B} \tilde{d}_j^m \nabla\psi_{x,j}^m \Big|^2 \\
& + \Big| \sum_{m=1}^{N_{jkl}} \tilde{b}_{jkl}^m \nabla\varphi_{jkl}^m + \sum_{m=1}^{N_{jkl}^B} \tilde{e}_k^m \nabla\psi_{y,k}^m + \tilde{g}^m \nabla\psi_{N_B}^m \Big|^2 + \Big| \sum_{m=1}^{N_{jkl}} \tilde{c}_{jkl}^m \nabla\varphi_{jkl}^m + \sum_{m=1}^{N_{jkl}^B} \tilde{f}_l^m \nabla\psi_{z,l}^m \Big|^2 d\mathbf{V} \\
\leq & \sum_{(j,k,l)\in S\setminus S_2} \left[N_{jkl} \sum_{m=1}^{N_{jkl}} \left(|\tilde{a}_{jkl}^m|^2 + |\tilde{b}_{jkl}^m|^2 + |\tilde{c}_{jkl}^m|^2 \right) \int_{Q_{jkl}} |\nabla\varphi_{jkl}^m|^2 d\mathbf{V} \right] \\
& + \sum_{(j,k,l)\in S_2} \Bigg[(N_{jkl} + N_{jkl}^B) \sum_{m=1}^{N_{jkl}} \left(|\tilde{a}_{jkl}^m|^2 + |\tilde{b}_{jkl}^m|^2 + |\tilde{c}_{jkl}^m|^2 \right) \int_{Q_{jkl}} |\nabla\varphi_{jkl}^m|^2 d\mathbf{V} \\
& + (N_{jkl} + N_{jkl}^B) \Bigg(\sum_{m=1}^{N_{jkl}^B} |\tilde{d}_j^m|^2 \int_{Q_{jkl}} |\nabla\psi_{x,j}^m|^2 + \sum_{m=1}^{N_{jkl}^B} |\tilde{e}_k^m|^2 \int_{Q_{jkl}} |\nabla\psi_{y,k}^m|^2 \\
& + \sum_{m=1}^{N_{jkl}^B} |\tilde{f}_l^m|^2 \int_{Q_{jkl}} |\nabla\psi_{z,l}^m|^2 + \sum_{m=1}^{N_{jkl}^B} |\tilde{g}^m|^2 \int_{Q_{jkl}} |\nabla\psi_{N_B}^m|^2 d\mathbf{V} \Bigg) \Bigg],
\end{aligned}
$$

where $S := \{(j,k,l) \in \mathbb{N}^3 : 1 \leq j \leq N_x, 1 \leq k \leq N_y, 1 \leq l \leq N_z\}$. By the quasi-regular property: $h/h_{\min} \leq \rho$, a direct calculation for the case of $DSSY$ bubble enrichment shows

$$
\begin{aligned}
\int_{Q_{jkl}} |\nabla\varphi_{jkl}^m|^2 d\mathbf{V} &= \frac{h_{y_k} h_{z_l}}{h_{x_j}} + \frac{h_{x_j} h_{z_l}}{h_{y_k}} + \frac{h_{x_j} h_{y_k}}{h_{z_l}} \leq 3h\rho \qquad \text{for } (j,k,l) \in S, \\
\int_{Q_{jkl}} |\nabla\psi_{x,j}^m|^2 d\mathbf{V} &= \frac{4687 h_{y_k} h_{z_l}}{3087 h_{x_j}} + \frac{400 h_{x_j} (h_{y_k}^2 + h_{z_l}^2)}{3087 h_{y_k} h_{z_l}} \\
&\leq h\rho \left(\frac{4687}{3087} + \frac{800}{3087}\rho \right) \qquad \text{for } (j,k,l) \in S_2,
\end{aligned} \tag{3.23}
$$

Similarly, $\psi^m_{y,k}$, $\psi^m_{z,l}$ and $\psi^m_{N_B}$ are bounded same quantity as $\psi^m_{x,j}$. Since each element Q_{jkl} has at most eight bases of vertex-type in $[\mathcal{NC}^h_0]^3$ and six bases of face-type in $\mathbf{B}^h$, we note that N_{jkl} and N^B_{jkl} are bounded by eight and six, respectively. Then by the boundedness of N_{jkl}, N^B_{jkl} and (3.23), we have

$$
\begin{aligned}
|\mathbf{v}_h|^2_{1,h} \le 14h\rho\left(\frac{4687}{3087} + \frac{800}{3087}\rho\right)\Bigg[&\sum_{(j,k,l)\in S}\sum_{m=1}^{N_{jkl}} (|\tilde{a}^m_{jkl}|^2 + |\tilde{b}^m_{jkl}|^2 + |\tilde{c}^m_{jkl}|^2) \\
+ \sum_{(j,k,l)\in S_2}\sum_{m=1}^{N^B_{jkl}} &(|\tilde{d}^m_j|^2 + |\tilde{e}^m_k|^2 + |\tilde{f}^m_l|^2 + |\tilde{g}^m|^2)\Bigg] \le C_1\|\underset{\sim}{a}\|^2,
\end{aligned}
$$

where $C_1 = C_1(h) := 112h\rho[(4687/3087) + (800/3087)\rho]$. Hence (3.22a) is shown.

By (3.20g), we get

$$
\begin{aligned}
\|q_h\|^2_{0,h} &= \sum_{(j,k,l)\in S\setminus S_1} q^2_{j,k,l} h_{x_j} h_{y_k} h_{z_l} + q^2_{j_0,k_0,l_0} h_{x_{j_0}} h_{y_{k_0}} h_{z_{l_0}} \\
&= \sum_{(j,k,l)\in S\setminus S_1} q^2_{j,k,l} h_{x_j} h_{y_k} h_{z_l} + (q_{j_0,k_0+1,l_0}|F_{j_0,l_0}| + g)^2 \frac{h_{y_{k_0}}}{h_{x_{j_0}} h_{z_{l_0}}}.
\end{aligned}
$$

Since $(a+b)^2 \le 2a^2 + 2b^2$ for $a, b \in \mathbb{R}$, one has $(q_{j_0,k_0+1,l_0}|F_{j_0,l_0}| + g)^2 \le 2h^4 q^2_{j_0,k_0+1,l_0} + 2g^2$. Applying this result and the quasi-regular property: $h/h_{\min} \le \rho$, we have

$$
\begin{aligned}
\|q_h\|^2_{0,h} &\le h^3 \sum_{(j,k,l)\in S\setminus S_1} q^2_{j,k,l} + 2\rho^2 h^3 q^2_{j_0,k_0+1,l_0} + 2h^{-1}\rho^2 g^2 \\
&\le (2\rho^2+1)h^3\|\underset{\sim}{q}\|^2 + 2h^{-1}\rho^2\|\underset{\sim}{a}\|^2.
\end{aligned}
$$

Then by (3.21), the inequality (3.22b) is shown for $C_2 = C_2(h) := (2\rho^2 + 1)[\lambda_1(h)]^{-1}h^3 + 2h^{-1}\rho^2$.

Combining (3.22a) and (3.22b), in view of (3.5) we have

$$b_h(\mathbf{v}_h, q_h) = \|\underset{\sim}{a}\|^2 \geq \beta(h)|\mathbf{v}_h|_{1,h}\|q_h\|_{0,h},$$

where $\beta(h) := 1/\sqrt{C_1(h)C_2(h)}$. Thus we complete the proof. $\square$

3.4 Numerical results

Now, we provide a numerical examples for the stationary Stokes problem on hexahedral meshes. In our numerical experiments, we consider the case that $\Omega = (0,1)^3$ is a unit cube and try to test two cases which are different positions of the bubble function which located at boundary and center in x-, y- and z-directions.

Example 1. For the first numerical example for the Stokes equations, the exact solution for velocity $\mathbf{u}(x,y,z)$, which is divergence-free, is given by $\nabla \times [1 \;\; 1 \;\; 1]^T\psi$, where

$$\psi(x,y,z) = \exp(x + 2y + 3z)x^2(x-1)^2y^2(y-1)^2z^2(z-1)^2,$$

with the exact solution for pressure $p(x,y,z)$ given by

$$\cos(\pi x)\cos(\pi y)\cos(\pi z).$$

Then the viscosity $\nu = 1$ and the body force term $\mathbf{f}$ can be generated by $-\Delta\mathbf{u} + \nabla p$. The numerical results are presented in Table 3.1 and Table 3.2 in terms of the discrete H^1-seminorm and L^2-norm convergence rates with bubble-vectors by the manner described in the subsection above. DOFs means the number of degrees of freedom.

meshes	DOFs	$\|u-u_h\|_0$	ratio	$\lvert u-u_h\rvert_{1,h}$	ratio	$\|p-p_h\|_0$	ratio
$4\times4\times4$	154	7.9622E-03	-	1.2981E-01	-	2.1122E-01	-
$8\times8\times8$	1562	2.4724E-03	1.68	7.7172E-02	0.75	8.7479E-02	1.27
$16\times16\times16$	14266	6.4380E-04	1.94	4.0446E-02	0.95	3.8844E-02	1.17
$32\times32\times32$	122234	1.6181E-04	1.99	2.0146E-02	0.99	1.8345E-02	1.08
$64\times64\times64$	1012474	4.0742E-05	1.99	1.0074E-02	1.00	8.9207E-03	1.04
$128\times128\times128$	8242682	1.0274E-05	1.99	5.0339E-03	1.00	4.3989E-03	1.02

Table 3.1: Error table of first example where bubble functions are located at boundary.

meshes	DOFs	$\|u-u_h\|_0$	ratio	$\lvert u-u_h\rvert_{1,h}$	ratio	$\|p-p_h\|_0$	ratio
$4\times4\times4$	154	8.0335E-03	-	1.2911E-01	-	1.9856E-01	-
$8\times8\times8$	1562	2.4661E-03	1.70	7.6831E-02	0.75	8.7356E-02	1.18
$16\times16\times16$	14266	6.4330E-04	1.94	4.0006E-02	0.94	3.8874E-02	1.17
$32\times32\times32$	122234	1.6181E-04	1.99	2.0145E-02	0.99	1.8347E-02	1.08
$64\times64\times64$	1012474	4.0742E-05	1.99	1.0074E-02	1.00	8.9207E-03	1.04
$128\times128\times128$	8242682	1.0274E-05	1.99	5.0339E-03	1.00	4.3999E-03	1.02

Table 3.2: Error table of first example where bubble functions are located at center.

Example 2. For the second numerical example, the exact solution for velocity $\mathbf{u}(x, y, z)$ is same solution from the **Example 1.**, with the exact solution for pressure $p(x, y, z)$ given by

$$\sin(2\pi x)\sin(2\pi y)\sin(2\pi z).$$

The numerical results are presented in Table 3.3 and Table 3.4 in terms of discrete H^1-seminorm and L^2-norm convergence rates with bubble-vectors by the manner described in the subsection above.

meshes	DOFs	$\|u - u_h\|_0$	ratio	$\|u - u_h\|_{1,h}$	ratio	$\|p - p_h\|_0$	ratio
$4 \times 4 \times 4$	154	8.1124E-03	-	1.3195E-01	-	2.7195E-01	-
$8 \times 8 \times 8$	1562	2.3707E-03	1.77	7.5877E-02	0.80	1.3844E-01	0.97
$16 \times 16 \times 16$	14266	6.2685E-04	1.92	3.9624E-02	0.94	6.9416E-02	1.00
$32 \times 32 \times 32$	122234	1.5947E-04	1.98	2.0038E-02	0.98	3.4715E-02	1.00
$64 \times 64 \times 64$	1012474	4.0409E-05	1.98	1.0047E-02	1.00	1.7357E-02	1.00
$128 \times 128 \times 128$	8242682	1.0321E-05	1.97	5.0272E-03	1.00	8.6781E-03	1.00

Table 3.3: Error table of second example where bubble functions are located at boundary.

meshes	DOFs	$\|u - u_h\|_0$	ratio	$\|u - u_h\|_{1,h}$	ratio	$\|p - p_h\|_0$	ratio
$4 \times 4 \times 4$	154	7.8485E-03	-	1.2961E-01	-	2.6886E-01	-
$8 \times 8 \times 8$	1562	2.3681E-03	1.73	7.5751E-02	0.77	1.3836E-01	0.96
$16 \times 16 \times 16$	14266	6.2686E-04	1.92	3.9619E-02	0.94	6.9417E-02	1.00
$32 \times 32 \times 32$	122234	1.5947E-04	1.97	2.0038E-02	0.98	3.4715E-02	1.00
$64 \times 64 \times 64$	1012474	4.0409E-05	1.98	1.0047E-02	1.00	1.7357E-02	1.00
$128 \times 128 \times 128$	8242682	1.0321E-05	1.98	5.0272E-03	1.00	8.6781E-03	1.00

Table 3.4: Error table of second example where bubble functions are located at center.

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국문초록

본 논문에서는 2차원과 3차원에서의 스토크스 방정식을 최소한의 자유도를 가지고 풀수 있는 유한요소법에 대해 소개한다. 각 사각형 또는 육면체에서 일차 다항식만으로 정의된 유한요소는 스토크스 방정식의 속도에 대한 수치적 접근에 적용된다. 동시에 스토크스 문제에서의 압력은 각 사각형 또는 육면체에서 조각 상수 함수로 근사 된다. 스토크스 문제에서 가장 중요한 inf-sup 조건을 만족함을 증명하고, 새롭게 제안한 유한요소들을 이용하여 2차원과 3차원 에서의 스토크스 방정식에 대한 수치적 결과를 보여줌으로써 이론적 결과를 뒷받침 한다.

2차원에서 이에 대한 응용으로 스토크스-다시-브링크만 인터페이스 문제에 제시한 유한요소를 적용해 본다. 스토크스-브링크만 인터페이스 문제에서는 제시한 유한요소를 이용하고, 스토크스-다시 인터페이스 문제에서는 제시한 유한요소와 라비아-토마 유한요소를 함께 이용한다. 이 또한 지금까지 알려진 유한요소들중에 가장 적은 자유도를 가지고 문제를 해결한다. inf-sup 조건과 수학적 오차분석을 해내고 이에 대한 수치적 결과를 보여줌으로써 이론적 결과를 뒷받침 한다.

주요어: 비순응 유한요소, P_1 비순응 유한요소, $DSSY$ 유한요소, inf-sup 조건, 비압축 스토크스 방정식, 스토크스-다시-브링크만 인터페이스 문제

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